Contact process with random slowdowns : phase transition and hydrodynamic limits

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Processus de contact avec ralentissements aléatoires
transition de phase et limites hydrodynamiques

Contact process with random slowdowns
phase transition and hydrodynamic limits

sous la direction d'**Ellen SAADA**

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À mes parents,
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"Would you tell me, please, which way I ought to go from here?"
"That depends a good deal on where you want to get to," said the Cat.
"I don't much care where—" said Alice.
"Then it doesn't matter which way you go," said the Cat.
"—so long as I get somewhere*," Alice added as an explanation.*
"Oh, you're sure to do that," said the Cat, "if you only walk long enough."

– Lewis Carroll, *Alice in Wonderland, Chapter VI.*

Résumé

Dans cette thèse, on étudie un système de particules en interaction qui généralise un processus de contact, évoluant en environnement aléatoire. Le processus de contact peut être interprété comme un modèle de propagation d'une population ou d'une infection. La motivation de ce modèle provient de la biologie évolutive et de l'écologie comportementale via la *technique du mâle stérile*, il s'agit de contrôler une population d'insectes en y introduisant des individus stérilisés de la même espèce : la progéniture d'une femelle et d'un individu stérile n'atteignant pas de maturité sexuelle, la population se voit réduite jusqu'à potentiellement s'éteindre.

Pour comprendre ce phénomène, on construit un modèle stochastique spatial sur un réseau dans lequel la population suit un processus de contact dont le taux de croissance est ralenti en présence d'individus stériles, qui forment un environnement aléatoire dynamique.

Une première partie de ce document explore la construction et les propriétés du processus sur le réseau $\mathbb{Z}^d$. On obtient des conditions de monotonie afin d'étudier la survie ou la mort du processus. On exhibe l'existence et l'unicité d'une transition de phase en fonction du taux d'introduction des individus stériles. D'autre part, lorsque $d = 1$ et cette fois en fixant l'environnement aléatoire initialement, on exhibe de nouvelles conditions de survie et de mort du processus qui permettent d'expliciter des bornes numériques pour la transition de phase.

Une seconde partie concerne le comportement macroscopique du processus en étudiant sa limite hydrodynamique lorsque l'évolution microscopique est plus complexe. On ajoute aux naissances et aux morts des déplacements de particules. Dans un premier temps sur le tore de dimension d, on obtient à la limite un système d'équations de réaction-diffusion. Dans un second temps, on étudie le système en volume infini sur $\mathbb{Z}^d$, et en volume fini, dans un cylindre dont le bord est en contact avec des réservoirs stochastiques de densités différentes. Ceci modélise des phénomènes migratoires avec l'extérieur du domaine que l'on superpose à l'évolution. À la limite on obtient un système d'équations de réaction-diffusion, auquel s'ajoutent des conditions de Dirichlet aux bords en présence de réservoirs.

Mots-clefs. système de particules en interaction, modèle stochastique spatial, processus de contact, milieu aléatoire, attractivité, percolation, transition de phase, limite hydrodynamique, réservoirs.

Abstract

In this thesis, we study an interacting particle system that generalizes a contact process, evolving in a random environment. The contact process can be interpreted as a spread of a population or an infection. The motivation of this model arises from behavioural ecology and evolutionary biology via the *sterile insect technique* ; its aim is to control a population by releasing sterile individuals of the same species : the progeny of a female and a sterile male does not reach sexual maturity, so that the population is reduced or potentially dies out.

To understand this phenomenon, we construct a stochastic spatial model on a lattice in which the evolution of the population is governed by a contact process whose growth rate is slowed down in presence of sterile individuals, shaping a dynamic random environment.

A first part of this document investigates the construction and the properties of the process on the lattice $\mathbb{Z}^d$. One obtains monotonicity conditions in order to study the survival or the extinction of the process. We exhibit the existence and uniqueness of a phase transition with respect to the release rate. On the other hand, when $d = 1$ and now fixing initially the random environment, we get further survival and extinction conditions which yield explicit numerical bounds on the phase transition.

A second part concerns the macroscopic behaviour of the process by studying its hydrodynamic limit when the microscopic evolution is more intricate. We add movements of particles to births and deaths. First on the d-dimensional torus, we derive a system of reaction-diffusion equations as a limit. Then, we study the system in infinite volume in $\mathbb{Z}^d$, and in a bounded cylinder whose boundaries are in contact with stochastic reservoirs at different densities. As a limit, we obtain a non-linear system, with additionally Dirichlet boundary conditions in bounded domain.

Keywords. interacting particle system, spatial stochastic model, contact process, random environment, attractiveness, percolation, phase transition, hydrodynamic limit, reservoirs.

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Introduction

Contents

This thesis examines two different aspects of a generalized contact process. In a microscopic scale, we study survival or extinction of the process with respect to varying parameters. Then, we go to a macroscopic scale and establish hydrodynamic limits, where in the dynamics of the underlying process we add displacements of particles and further on migratory phenomena.

In this chapter, we introduce some general settings we shall make use of, first on interacting particle systems in Section 1.1 and then on the contact process in Section 1.2. After what, in Section 1.4, we develop shortly the big picture of the sterile insect technique. In Section 1.5, we describe a generalized contact process and our results that lead to an understanding of this competition model.

1.1 Interacting Particle Systems

Interacting particle systems are a class of Markov processes that arose in the early seventies due to pioneering works by F. Spitzer [70, 71] and R.L. Dobrushin [16]. They have provided a framework that describes the space-time evolution of an infinity of indistinguishable particles governed by a strong random and local interaction.

This particular class of stochastic processes comes up in various areas of applications : physics, biology, computer science, economics and sociology,... that dictate the nature of the randomness of the processes.

1.1.1 The setup

As a preparation, one first reviews some necessary background theory about interacting particle systems. For further contents on the topic, one refers the reader to T.M. Liggett's books [58, 57].

State spaces are of the form $\Omega = F^S$, where F is discrete and finite, S is a countable set of sites. Note that Ω is compact in the product topology. A *configuration* $\zeta \in \Omega$ is described by the *state* of each site x of the graph S, given at time t by $\zeta_t(x) \in F$. For each $\zeta \in \Omega$ and $T \subset S$, the local dynamics of the system is depicted by a collection of *transition measures* $c_T(\zeta, d\alpha)$, assumed to be finite and positive on F^T. Assume further that the mapping $\zeta \mapsto c_T(\zeta, d\alpha)$ is continuous from Ω to the space of finite measures on F^T with the topology of weak convergence. If ζ is the current configuration, a transition of state or *flip* involving the coordinates in T occurs at rate $c_T(\zeta, F^T)$ and $c_T(\zeta, d\alpha)/c_T(\zeta, F^T)$ is the distribution of the resulted configuration restricted to T.

We will use the notation $\mathbb{P}^\zeta$ for the distribution of the process $(\zeta_t)_{t\geqslant 0}$ starting from the initial configuration ζ, and $\mathbb{E}^\zeta$ will denote the corresponding expectation. The infinitesimal description of a process $\zeta \in \Omega$ is given by its *generator* $\mathcal{L}$, a linear unbounded operator defined on an appropriate dense domain $\mathscr{D}(\Omega)$ of the space of functions $f : \Omega \to \mathbb{R}$. For any cylinder function f, i.e. that depends only on finitely many coordinates, $\mathcal{L}$ is defined by

$$\mathcal{L}f(\zeta) = \sum_T \int_\Omega c_T(\zeta, d\alpha)\big(f(\zeta^\alpha) - f(\zeta)\big), \tag{1.1.1}$$

where ζ^α is obtained from ζ only by flipping the coordinates in T, that is, for $\alpha \in F^T$,

$$\zeta^\alpha = \begin{cases} \zeta(x) \text{ if } x \notin T, \\ \alpha(x) \text{ if } x \in T. \end{cases}$$

The series converges provided that $c_T(.,.)$ satisfies natural summability conditions.

Let $C(\Omega)$ be the space of continuous real-valued functions on Ω equipped with the uniform norm. All the processes we consider here have the Feller property (i.e. strong Markov processes whose transition measures are weakly continuous in the initial state) so that the *semigroup* S_t of the process on $C(\Omega)$ is well defined :

Theorem 1.1.1. *Suppose $\{S_t, t > 0\}$ is a Markov semigroup on $\mathcal{C}(\Omega)$. Then there exists a unique Markov process $\{P^\zeta, \zeta \in \Omega\}$ such that*

$$S_t f(\zeta) = \mathbb{E}^\zeta f(\zeta_t)$$

for all $f \in \mathcal{C}(\Omega)$, $\zeta \in \Omega$ and $t \geqslant 0$.

The link binding the infinitesimal description of the process (generator) to the time-evolution of the process (semigroup) is given by the Hille-Yosida theory set in the Banach space $C(\Omega)$.

Theorem 1.1.2 (Hille-Yosida)**.** *There is a one-to-one correspondence between Markov generators on $C(\Omega)$ and Markov semigroups on $C(\Omega)$. This correspondence is given by*

1. $\mathscr{D}(\Omega) = \left\{ f \in C(\Omega) : \lim_{t \downarrow 0} \frac{S_t f - f}{t} \text{ exists} \right\}$, *and*

$$\mathcal{L}f = \lim_{t \downarrow 0} \frac{S_t f - f}{t}, \ \ f \in \mathscr{D}(\Omega).$$

2. *for $t \geqslant 0$,*

$$S_t f = \lim_{n \to \infty} (f - \frac{t}{n}\mathcal{L}f)^{-n}, \ \ f \in C(\Omega).$$

Relying on the Hille-Yosida theory, the following result states sufficient conditions for the existence of an infinite particle system.

Theorem 1.1.3 (T.M. Liggett (1972))**.** *Assume that*

$$\sup_{x \in S} \sum_{T \ni x} \sup \Big(c_T(\zeta, F^T) : \zeta \in \Omega \Big) < \infty$$

and

$$\sup_{x \in S} \sum_{T \ni x} \sum_{u \neq x} \sup \Big(\|c_T(\zeta_1, d\alpha) - c_T(\zeta_2, d\alpha)\|_T : \zeta_1(y) = \zeta_2(y) \text{ for all } y \neq u \Big) < \infty$$

where $\| \cdot \|_T$ stands for the total variation norm of a measure on F^T. Then the closure $\overline{\mathcal{L}}$ of $\mathcal{L}$ defined in (1.1.1) *is the generator of a Feller Markov process $(\zeta_t)_{t \geqslant 0}$ on Ω. In particular, if f is a cylinder function then,*

$$\mathcal{L}f = \lim_{t \to 0} \frac{S_t f - f}{t},$$
$$\overline{\mathcal{L}} S_t f = S_t \mathcal{L} f$$

and $u(t) = S_t f$ is the unique solution to the evolution equation

$$\partial_t u(t) = \overline{\mathcal{L}} u(t), \ \ u(0) = f. \tag{1.1.2}$$

Let $\mathfrak{P}$ be the set of probability measures on Ω equipped with the topology of weak convergence, i.e.

$$\mu_n \to \mu \text{ in } \mathfrak{P} \text{ if and only if } \int_\Omega f d\mu_n \to \int_\Omega f d\mu$$

for all $f \in C(\Omega)$. Note that the compactness of Ω implies the compactness of $\mathfrak{P}$ in this induced topology.

1.1.2 Invariant measures

Study of interacting particle systems involves use of their invariant measures and ideally, convergence to them. If μ is a probability measure on Ω, the distribution of ζ_t with initial distribution μ is denoted by μS_t and is defined by

$$\int_\Omega f d(\mu S_t) = \int_\Omega S_t f d\mu, \; f \in C(\Omega).$$

By the Riesz Representation theorem, this relation defines uniquely μS_t. The measure μ is *invariant* with respect to the process if $\mu S_t = \mu$ for all $t > 0$. Denote by $\mathfrak{I}$ the set of all invariant measures. Furthermore,

Theorem 1.1.4 (Proposition 1.8 [58]). *i.* $\mu \in \mathfrak{I}$ *if and only if*

$$\int_\Omega \mathcal{L} f d\mu = 0, \; \textit{for all cylinder functions } f.$$

ii. $\mathfrak{I}$ *is compact, convex and non-empty.*

iii. $\mathfrak{I}$ *is the closed convex hull of its extreme points.*

iv. Let $\mu \in \mathfrak{P}$. *If* $\overline{\mu} := \lim_{t\to\infty} \mu S_t$ *exists, then* $\overline{\mu} \in \mathfrak{I}$.

Remark that a process always has at least one invariant measure. This measure might satisfy a symmetry property called *reversibility* that allows simpler computations or even, further results. A probability measure μ on Ω is *reversible* for the process if

$$\int_\Omega f S_t g d\mu = \int_\Omega g S_t f d\mu, \; \text{for all } f, g \in C(\Omega)$$

or equivalently,

$$\int_\Omega f \mathcal{L} g d\mu = \int_\Omega g \mathcal{L} f d\mu, \; \text{for all cylinder functions } f, g.$$

1.1.3 Coupling and stochastic order

A coupling is a construction of two (or even more) stochastic processes on a common probability space. To make use of this powerful tool, we will deal with several topics that are closely connected with coupling such as stochastic order relations between probability measures, monotone processes and correlation inequalities. These useful relations allow us to compare processes, so that one can deduce properties from one to another by domino effect.

Assuming that F is totally ordered, the state space Ω is a *partially ordered set*, with partial order given by

$$\zeta \leqslant \zeta' \text{ if } \; \text{for all } x \in S, \; \zeta(x) \leqslant \zeta'(x), \tag{1.1.3}$$

where this last inequality refers to the order on F. A function $f \in C(\Omega)$ is *increasing* if

$$\zeta \leqslant \zeta' \Rightarrow f(\zeta) \leqslant f(\zeta').$$

This leads naturally to define the stochastic order between two probability measures μ_1 and μ_2 on Ω, that is, μ_2 is *stochastically larger* than μ_1, written $\mu_1 \leqslant \mu_2$ if :

$$\int_\Omega f d\mu_1 \leqslant \int_\Omega f d\mu_2 \text{ for any increasing function } f \text{ on } \Omega.$$

A necessary and sufficient condition for a semigroup, acting on measures, to preserve the ordering on Ω is given by

Theorem 1.1.5 (Theorem 2.2 [57]). *For a Feller process on Ω with semigroup S_t, the following two statements are equivalent :*

a. If f is an increasing function on Ω then $S_t f$ is an increasing function of Ω for all $t \geqslant 0$.

b. If $\mu_1 \leqslant \mu_2$ then $\mu_1 S_t \leqslant \mu_2 S_t$ for all $t \geqslant 0$.

Stochastic order between two particle systems $(\zeta_t)_{t\geqslant 0}$ and $(\zeta'_t)_{t\geqslant 0}$ is given by the existence of a coupled process $(\zeta_t, \zeta'_t)_{t\geqslant 0}$ on the probability space $\Omega \times \Omega$ that preserves the order between their initial configurations, that is, if $\zeta_0 \leqslant \zeta'_0$ then $\zeta_t \leqslant \zeta'_t$ a.s. for all $t > 0$. Such a coupling is said to be increasing and ζ'_t is said to be stochastically larger than ζ_t. When $(\zeta_t)_{t\geqslant 0}$ and $(\zeta'_t)_{t\geqslant 0}$ are two copies of the same process, we say the process is *attractive*.

The following result gives the connection between coupling and stochastic order.

Theorem 1.1.6 (Theorem 2.4 [58]). *Let μ_1 and μ_2 be probability measures on Ω. Then μ_2 is stochastically larger than μ_1 if and only if there exists a coupling (ζ, ζ') such that ζ has distribution μ_1, ζ' has distribution μ_2 and $\zeta \leqslant \zeta'$ almost surely, that is, there exists a measure ν on Ω such that*

$$\begin{aligned} \nu\{(\zeta, \zeta') : \zeta \in A\} &= \mu_1(A) \\ \nu\{(\zeta, \zeta') : \zeta' \in A\} &= \mu_2(A) \\ \nu\{(\zeta, \zeta') : \zeta \leqslant \zeta'\} &= 1 \end{aligned}$$

Furthermore, we will consider different types of stochastic processes :

$(\xi_t)_{t\geqslant 0}$	(basic) contact process
$(\xi_t, \omega_t)_{t\geqslant 0}$	contact process in dynamic random environment
$(\eta_t)_{t\geqslant 0}$	multitype contact process

1.2 A short story of the contact process

Introduced by T.E. Harris in 1974 [39], the *contact process on the graph S with growth rate λ_1* is an interacting particle system $(\xi_t)_{t\geqslant 0}$ on $\{0,1\}^S$, whose dynamics is given by the following transition measure : the involved sets T are singletons $T = \{x\}$ and,

$$c_T(\xi, d\alpha) = \begin{cases} \lambda_1 n_1(x,\xi)\delta_{\{1\}} & \text{if } \xi(x) = 0, \\ \delta_{\{0\}} & \text{if } \xi(x) = 1, \end{cases} \tag{1.2.1}$$

where $n_i(x,\xi) = \sum_{y\in S:|y-x|=1} \mathbf{1}\{\xi(y) = 1\}$ stands for the number of neighbours of site x that are in state i. Here $|\cdot|$ refers to the maximum norm : $|x| = \max_{1\leqslant j\leqslant d} |x_j|$, for $x \in \mathbb{R}^d$. Denote by $\mathbb{P}_{\lambda_1}$ the law of the contact process with growth rate λ_1.

It is usually interpreted as the spread of a population, an infection or a rumour. Regarded as an infection, infected sites (in state 1) become healthy spontaneously after a unit exponential time while healthy sites (state 0) become infected at some rate, proportional to the number of their infected neighbours.

General theory about the contact process is finely exposed by T.M Liggett [58] for results from 1974 to 1985, [57] for results after 1985 and by R. Durrett [18] as well.

1.2.1 Construction of the process

Let A be a subset of S. Define ξ_t^A as the process starting from the initial configuration $\xi_0 = \mathbf{1}_A$. Configurations $\xi \in \{0,1\}^S$ are commonly identified with subsets of S via

$$\Xi_t^A = \{x \in S : \xi_t^A(x) = 1\},$$

regarded as the set of occupied sites at time t. When $A = \{0\}$, we will omit the exponent. As a consequence of Theorem 1.1.3, the transition measure $c_T(\xi, d\alpha)$ uniquely defines a Markov process, so that the infinitesimal generator of the contact process is defined for any cylinder function f on $\{0,1\}^S$ by

$$\mathcal{L}_1 f(\xi) = \sum_{x\in S} \int_\Omega c_T(\xi, d\alpha)[f(\xi^\alpha) - f(\xi)] \tag{1.2.2}$$

Graphical representation The graphical construction of the contact process is due to T.E. Harris [40]. The idea is to construct a percolation structure on which to define the process, lending itself to the use of the theory of percolation (see G. Grimmett [33]). To carry out this representation, for each pair $(x,y) \in S^2$ that are joined by an edge in S, let $\{T_n^{x,y}, n \geqslant 1\}$ be the arrival times of independent rate λ_1 Poisson processes and for each $x \in S$, let $\{D_n^x, n \geqslant 1\}$ be the arrival times of independent rate 1 Poisson processes.

Both families of Poisson processes are mutually independent. Now, think of the space-time diagram $S \times [0, \infty)$. At time $t = D_n^x$, put a death symbol "x" at $(x, t) \in S \times [0, \infty)$. At time $t = T_n^{x,y}$, draw an arrow from (x, t) to (y, t).

By way of illustration, see Figure 1.1.

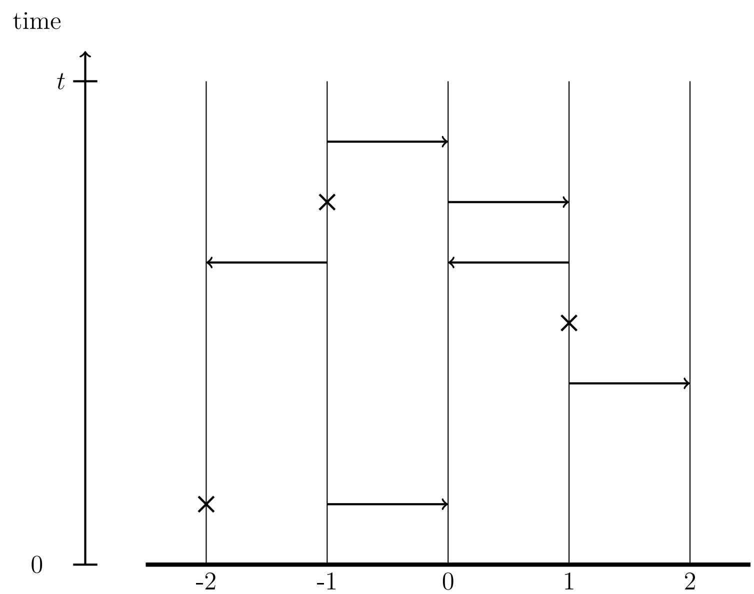

FIGURE 1.1: The graphical representation for the contact process on $\mathbb{Z}^1 \times \mathbb{R}_+$

For $s \leqslant t$, there exists an *active path* in the space-time picture $S \times [0, \infty)$ from (x, s) to (y, t), written $(x, s) \to (y, t)$, if there exists a sequence of times $s = s_0 < s_1 < ... < s_{n-1} < s_n = t$ and spatial locations $x = x_0, x_1, ..., x_n = y$ such that

i. for $i = 1, ..., n$, there is an arrow from x_{i-1} to x_i at time s_i.

ii. for $i = 0, ..., n-1$, the vertical segments $\{x_i\} \times (s_i, s_{i+1})$ contain no death symbol.

In words, an active path is a connected oriented path that moves forward in time without crossing a death symbol and along the directions of the arrows. For instance, in Figure 1.1, there is an active path from $(0, 0)$ to $(1, t)$. The contact process with initial configuration $A \subset S$ is obtained by setting

$$A_t^A := \{y \in S : \exists x \in A \text{ such that } (x, 0) \to (y, t)\}$$

Therefore, in our previous example, $A_t^{\{0\}} = \{1\}$.

The graphical construction provides a joint coupling of contact models with different transition rates : let $\lambda_1^{(1)} \leqslant \lambda_1^{(2)}$, if we constructed the process with rate $\lambda_1^{(2)}$ and we keep each arrow with probability $\lambda_1^{(1)}/\lambda_1^{(2)}$, by the thinning property of the Poisson processes, we end up with the graphical representation of a contact process with growth rate $\lambda_1^{(1)}$.

Thus, one has a non-decreasing growth with respect to λ_1. On the other hand, it also provides a monotone coupling :

$$A \subset B \Rightarrow A_t^A \subset A_t^B,$$

Therefore, the contact process is attractive and it also follows from the graphical construction that the contact process is additive (see D. Griffeath [32]) :

$$A_t^{A \cup B} = A_t^A \cup A_t^B.$$

1.2.2 Upper invariant measure and duality

Since the partial order on Ω defined in (1.1.3) induces one on the set of probability measures on Ω, there will be a lowest and largest element on $\mathfrak{I}$ with respect to this partial order.

If $\underline{0}$ denotes the configuration identically equal to 0, since 0 is an absorbing state then $\delta_{\underline{0}}$ is called the *lower invariant measure* for the contact process. The upper invariant measure can be constructed using attractiveness : choose the initial configuration as the biggest possible one, i.e. starting from $\Xi_0 = S$, and let μ_t be the distribution of ξ_t, so that $\mu_0 = \delta_{\underline{1}}$. Then $\mu_t \leqslant \mu_0$. By attractiveness and applying the Markov property, we have $\mu_{t+s} \leqslant \mu_t$ for all $s > 0$. Therefore, $t \mapsto \mu_t$ is decreasing and in particular, for every increasing function f on Ω, the map $t \mapsto \int_\Omega f d\mu_t$ is decreasing as well. Since $\mathfrak{P}(\Omega)$ is compact for the weak topology, the limiting distribution

$$\overline{\mu} := \lim_{t \to \infty} \mu_t$$

exists and is the upper invariant measure of the process. It is invariant as a limiting measure for the Markov process by Theorem 1.1.4. In particular, the measure $\overline{\mu}$ has positive correlations.

Correlation inequalities will be crucial property in Section 2.6 where we will work in arbitrary large but finite spaces. A probability measure μ on Ω has *positive correlations* if

$$\int_\Omega fg d\mu \geqslant \int_\Omega f d\mu \int_\Omega g d\mu,$$

for all increasing functions f, g on Ω. A sufficient condition for a measure to have positive correlations is given by the following result.

Theorem 1.2.1 (C. Fortuin, P. Kasteleyn and J. Ginibre [29]). *Suppose S is* finite. *Let μ be a probability measure on Ω such that for all $\zeta, \zeta' \in X$*

$$\mu_1(\max(\zeta, \zeta'))\mu_2(\min(\zeta, \zeta')) \geqslant \mu_1(\zeta)\mu_2(\zeta')$$

Then μ has positive correlations.

One essential property satisfied by the contact process is that it is self-dual [34, Proposition 6.5], that is, the dual process is again a contact process. For $A, B \subset S$,

$$\mathbb{P}_{\lambda_1}(\Xi_t^A \cap B \neq \varnothing) = \mathbb{P}_{\lambda_1}(\Xi_t^B \cap A \neq \varnothing) \tag{1.2.3}$$

This property allows us to link an equality relation between survival probability and density of 1's under the upper invariant measure. Indeed, since $\{\Xi_{t+1}^{\{0\}} \cap S \neq \varnothing\} \subset \{\Xi_t^{\{0\}} \cap S \neq \varnothing\}$ for all $t \geqslant 0$, $t \mapsto \{\Xi_t^{\{0\}} \cap S \neq \varnothing\}$ is non-increasing,

$$\lim_{t\to\infty} \mathbb{P}_{\lambda_1}(\Xi_t^{\{0\}} \cap S \neq \varnothing) = \mathbb{P}_{\lambda_1}(\forall t \geqslant 0,\ \Xi_t^{\{0\}} \neq \varnothing)$$

By self-duality, applying (1.2.3) with $A = \{0\}$ and $B = S$, one obtains

$$\mathbb{P}_{\lambda_1}(\Xi_t^{\{0\}} \cap S \neq \varnothing) = \mathbb{P}_{\lambda_1}(\Xi_t^S \cap \{0\} \neq \varnothing).$$

The right-hand side is $\mathbb{P}_{\lambda_1}(\Xi_t^S(0) = 1)$, and by weak convergence of μ_0 to $\overline{\mu}$, one has

$$\lim_{t\to\infty} \mathbb{P}_{\lambda_1}(\Xi_t^S(0) = 1) = \bar{\mu}\{\xi :\ \xi(0) = 1\}$$

where $\overline{\mu}$ stands for the upper invariant measure of $(\xi_t)_{t\geqslant 0}$. By translation invariance of $\overline{\mu}$,

$$\lim_{t\to\infty} \mathbb{P}_{\lambda_1}(\Xi_t^{\{0\}} \cap S \neq \varnothing) = \lim_{t\to\infty} \mathbb{P}_{\lambda_1}(\xi_t^S(0) = 1) = \bar{\mu}\{\xi :\ \xi(x) = 1\} \tag{1.2.4}$$

1.2.3 Survival and extinction

A key feature of the contact process lies in the fact its growth does not evolve spontaneously but depends on some neighbourhood. In words, the configuration $\underline{0}$ is a trap and a natural question is whether the individuals survive, that is, if there is infinitely often a site in state 1. The main feature of the contact process is that it exhibits a phase transition in the following way.

Define the survival event of the process by $\{\forall t \geqslant 0,\ \Xi_t \neq \varnothing\}$ with the initial configuration $\xi_0 = \mathbf{1}_{\{0\}}$. The contact process is said to die out if

$$\mathbb{P}_{\lambda_1}(\forall t \geqslant 0,\ \Xi_t \neq \varnothing) = 0$$

and to survive strongly if

$$\mathbb{P}_{\lambda_1}(\overline{\lim_{t\to\infty}}\, \xi_t(0) = 1) > 0.$$

The process is said to survive weakly if it survives but not strongly, that is,

$$\mathbb{P}_{\lambda_1}(\forall t \geqslant 0,\ \Xi_t \neq \varnothing) > 0.$$

Using these definitions and monotonicity, we are now ready to define the two following critical values :

$$\lambda_c = \inf\{\lambda_1 : \mathbb{P}_{\lambda_1}(\forall t \geqslant 0\ \Xi_t \neq \varnothing) > 0\} \tag{1.2.5}$$

and

$$\lambda_s = \inf\{\lambda_1 : \mathbb{P}_{\lambda_1}(\varlimsup_{t\to\infty} \xi_t(0) = 1) > 0\}. \tag{1.2.6}$$

for which, the process

$$\begin{array}{rl} \text{dies out if} & \lambda_1 < \lambda_c \\ \text{survives weakly if} & \lambda_c < \lambda_1 < \lambda_s \\ \text{survives strongly if} & \lambda_1 > \lambda_s \end{array}$$

Since

$$\{\varlimsup_{t\to\infty} \xi_t(0) = 1\} \subset \{\forall t \geqslant 0\ \Xi_t \neq \varnothing\},$$

if the process survives weakly then it survives strongly thus $\lambda_c \leqslant \lambda_s$.

On the $d-$dimensional integer lattice $\mathbb{Z}^d$, one of the most important results about the contact process is the existence and uniqueness of a critical value $\lambda_c = \lambda_s$.

Theorem 1.2.2 (T.E. Harris [39])**.** *There exists a critical value $\lambda_c \in (0, \infty)$ such that the contact process survives if $\lambda_1 > \lambda_c$ and dies out if $\lambda_1 < \lambda_c$, i.e.*

$$\mathbb{P}_{\lambda_1}(\forall t \geqslant 0,\ \Xi_t \neq \varnothing) = 0 \text{ if } \lambda_1 < \lambda_c,$$
$$\mathbb{P}_{\lambda_1}(\forall t \geqslant 0,\ \Xi_t \neq \varnothing) > 0 \text{ if } \lambda_1 > \lambda_c.$$

After having been an open question during about fifteen years, the critical behaviour has been given by

Theorem 1.2.3 (C. Bezuidenhout and G.R. Grimmett [5])**.** *The critical contact process dies out, that is,*

$$\mathbb{P}_{\lambda_c}(\forall t \geqslant 0,\ \Xi_t \neq \varnothing) = 0.$$

R. Holley and T.M. Liggett [41] proved $\lambda_c \leqslant 2$ in the one-dimensional case. An improved upper bound 1.942 was given by T.M. Liggett [54]. More generally, one has for the general case $d \geqslant 1$,

$$(2d-1)^{-1} \leqslant \lambda_c \leqslant 2d^{-1},$$

see N. Konno [47] for further information on bounds of the contact process.

1.3 Hydrodynamic limits

Hydrodynamic limits are a device that arose in statistical physics to derive deterministic macroscopic evolution laws assuming the underlying microscopic dynamics are stochastic.

By way of illustration, consider the evolution of a system constituted of a large number of components (such as a fluid), one can examine and characterize the equilibrium states of the system through macroscopic quantities (such as temperature or pressure). Now, investigating the fluid in a volume which is small macroscopically but large microscopically, the system is close to an equilibrium state and characterized by some spatial parameter. As the local equilibrium picture should evolve in a smooth way, at some time t the system is close to a new equilibrium state now characterized by a parameter depending on space and time. This space-time parameter evolves smoothly in time according to a partial differential equation, the hydrodynamic equation.

To take the limit from the microscopic to the macroscopic system, we need to introduce a suitable space-time scaling. Consider a microscopic space S_N embedded in a corresponding macroscopic space S (e.g. $S_N = (\mathbb{Z}/N\mathbb{Z})^d$ and $S = (\mathbb{R}/\mathbb{Z})^d$) so each microscopic vertex $x \in S_N$ is associated to a macroscopic vertex $x/N \in S$. Therefore, distance between particles converges to zero. Besides, we renormalize the time by linking a microscopic time t to a macroscopic time $t\theta(N)$ (e.g. $\theta(N) = N^2$), since more time is needed in the macroscopic scale to observe movements of particles.

To investigate the hydrodynamic behaviour of interacting particle systems we shall prove that starting from a sequence of measures associated to some initial density profile ρ_0, in the following sense

$$\overline{\lim_{N\to\infty}} \mu_N \left(\left| \frac{1}{N^d} \sum_{x\in S_N} G(x/N)\eta(x) - \int_S G(u)\rho_0(u)du \right| > \delta \right) = 0 \tag{1.3.1}$$

for any $\delta > 0$ and continuous function $G : S \to \mathbb{R}$, then at some renormalized time $t\theta(N)$, we obtain a state $S_{t\theta(N)}\mu_N$ associated to a new density profile $\rho_t(\cdot)$ that is a weak solution of a partial differential equation. That is,

$$\overline{\lim_{N\to\infty}} \mu_N \left(\left| \frac{1}{N^d} \sum_{x\in S_N} G(x/N)\eta_{t\theta(N)}(x) - \int_S G(u)\rho_t(u)du \right| > \delta \right) = 0. \tag{1.3.2}$$

In other words, the sequence of measures μ_N integrates the density ρ_t at the macroscopic point $u \in S$ in the same way than an equilibrium measure of density $\gamma(u)$ does.

Since we shall work in a fixed space as N increases, we will examine the time-evolution of the empirical measures associated to the interacting particle system : for a configuration $\eta \in \Omega$, define the empirical measure $\pi^N(\eta)$ on S associated to η by

$$\pi^N(\eta) = N^{-d} \sum_{x\in S_N} \eta(x)\delta_{x/N}, \tag{1.3.3}$$

where δ_x represents the Dirac measure concentrated on x. This way, we can express (1.3.2) in terms of the empirical density, by integrating G with respect to π^N. Since there is a one-to-one correspondence between a configuration η and empirical measure $\pi^N(\eta)$, the measure π_t^N inherits the Markov property.

The goal to derive the hydrodynamic limits is to prove the empirical measure π_t^N converges in probability to an absolutely continuous measure $\rho(t,u)du$ where $\rho_t(u)$ is the solution of a partial differential equation with initial condition ρ_0.

Monographs dealing with hydrodynamic limits include A. De Masi and E. Presutti [15], H. Spohn [72] C. Kipnis and C. Landim [42].

1.4 From life and nature

During the last decades, a better understanding of biological phenomena has arisen the need to study stochastic spatial processes. Authors such as R. Durrett, R. Schinazi, or J. Schweinsberg have deemed the relation of interacting particle systems to biological, ecological and medical frameworks. A quick interesting overview may be found in joint papers of R. Durrett with the biologist S. Levin [21, 22], and [20].

In this document, the biological phenomenon we are concerned is the so-called *Sterile insect technique* (SIT). Due to entomologists R.C. Bushland and E.F. Knipling's works [46] in the fifties, it is a pest control method whereby sterile individuals of the population to either regulate or eradicate are released. While sterile males compete with wild males, they eventually mate with (wild) females preventing the apparition of progenies. By repeated releases, we should be able to cause a variety of outcomes ranging from reduction to extinction.

1.4.1 The sterile insect technique

In the thirties and forties, the idea of designing a gene that actively spreads through a pest population without conveying some fitness advantage had arisen independently by A. S. Serebrovskii (Moscow State University), F. L. Vanderplank (Bristol Zoo and Tanzania Research Department) and E. F. Knipling (United States Department of Agriculture). Serebrovskii and Vanderplank both sought to achieve pest control through partial sterility that occurs when different species or genetic strains were hybridized (using chromosomal translocations or crossing) : competition between two interbreeding strains doesn't favour the fitter group, involving the genetic property called *under-dominance* which can actually cause the strain with greater fitness to die out.

Discovery and first success story. Discovery of induced mutagenesis by 1946 Nobel Prize H.J. Muller conducted Bushland and Knipling to use ionizing radiation in the sterilization process to get rid of the *new world screw-worm fly (Cochliomyia hominivorax)*.

After successful eradication programs carried out in Curaçao and Florida in the late fifties, the technique was applied during the next decades to eradicate the screw-worm from the USA, Mexico, and Central America to Panama, until it has been declared a fly-free area.

The big picture. Food safety, quality and biodiversity have required demands at national and international levels for the development and introduction of area-wide (and biological approaches) for integrated management of pest control.

Fruit flies are a major interference in the marketing of fruit and vegetable commodities, preventing therefore important economic developments. The *Mediterranean fruit* *fly (medfly)* is a notorious insect pest threatening multi-million commodities export trade throughout the world.

In the seventies, a first large-scale program stopped the invasion of the medfly from Central America. Eradication from Mexico and maintaining the country free of this pest at an annual cost of US$ 8 million, has protected fruit and vegetable export markets of close to US$ 1 billion a year.

In Japan, the SIT was employed in the eighties and nineties to eradicate the melon fly in Okinawa and south-western islands, permitting access for fruits and vegetables produced in these islands to the main markets in the mainland. A program with Peru operates in Argentina, northern Chile and southern Peru. Chilean fruits have entered the US market for exports estimated to up to US$ 500 million per year.

More recently, the SIT is increasingly applied with eradication programs of fruit flies ongoing in Middle-East (Israel, Jordan, Palestine), South Africa, and Thailand ; in preparation in Brazil, Portugal, Spain, and Tunisia.

Economic benefits have been confirmed so that for medflies and other fruit flies, the current worldwide production capacity of sterile individuals has reached several billion a week.

Future trends. Lauded for its attributes in terms of economics, environment and safety, the technique has successfully been able to get rid of populations threatening livestocks, fruits, vegetables, and crops. But besides economic reasons to involve SIT, public health issues have induced governments to request supports from International Atomic Energy Agency (IAEA) and Food and Agriculture Organization of the United Nations (FAO) for SIT initiatives to stem vector-borne diseases.

1.4.2 Time to unleash the mozzies ?

Thinking about the deadliest animal in the world, mosquitoes would not hit our minds. But one estimates about 1 million people per year die from mosquito-borne diseases, such as malaria, dengue fever, etc ... [Source : World health organization].

Urbanisation, globalisation and climate change have accelerated the spread and increased the number of outbreaks of new mosquito-borne diseases, such as the dengue.

Considered as the fastest growing disease, dengue fever is currently not cured by any vaccine or effective antiviral drug, meaning that mosquito control is the only viable option to control the disease at short notice. The SIT has the potential to reduce the targeted mosquitoes population to a level below which the disease is not transmitted. A first trial using sterile mosquitoes was conducted in El Salvador in the seventies, where 4.4 million sterile individuals were released in a 15 square km area over 22 weeks, eradicating successfully the targeted population. Going on a much larger area, total suppress of the population failed due to an immigration of local mosquitoes into the trial area.

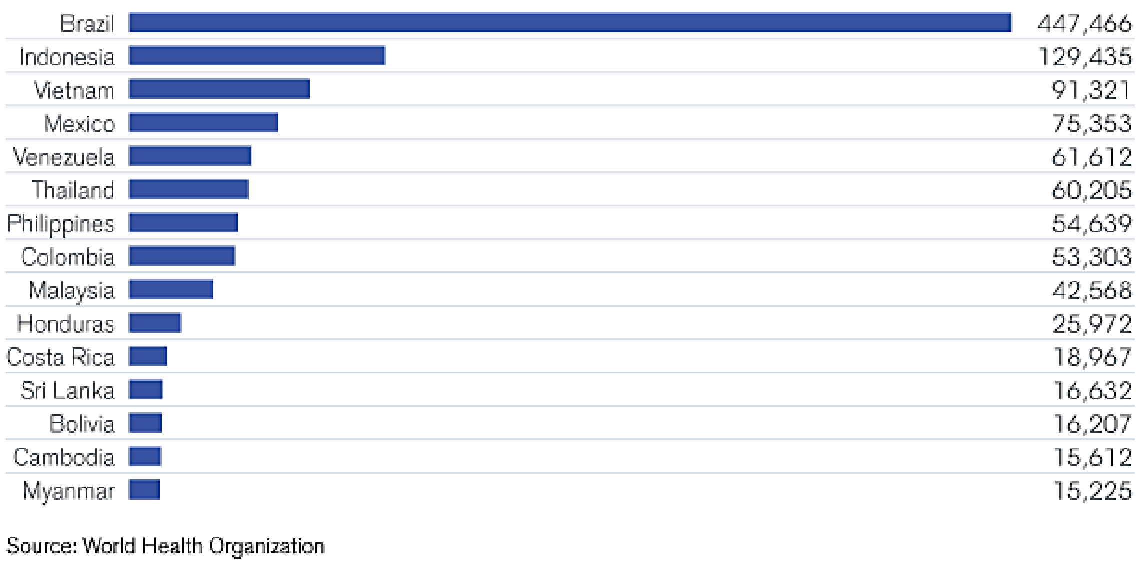

FIGURE 1.2: Average number of dengue cases in most highly endemic countries as reported to WHO 2004-2010.

Being the highest endemic country of dengue, the brazilian government is highly concerned by the expansion of the dengue fever. According to pilot-scale releases in the state of Bahia started in june 2013, releases of genetically modified mosquitoes resulted in a 96% reduction of the wild population in the target area after 6 months- level maintained for a further 7 months using continued releases, at reduced rates, to avoid re-infestation.

The National Technical Commission for Biosecurity (CTNBio) in Brazil recently approved (april 2014) the commercial release of genetically modified mosquitoes in a bid to curb outbreaks of dengue fever. As of july 2014, the research program in the state of Bahia is waiting for an approval granted by the Brazilian Health Surveillance Agency (ANVISA) to ensue a scaling-up of the program. [Source : Comissão Técnica Nacional de Biossegurança (CTNBio), Agência Nacional de Vigilância Sanitária (ANVISA).]

1.4.3 Past mathematical models

Even if models of population dynamics are typically posed as difference or differential equations, such as predator-prey systems (whose Nicholson-Bailey and Lotka-Volterra models are the work horses), stochastic models give additional information on the expected variability of the resulting control. Some of them were developed by Kojima (1971), Bogyo (1975), Costello and Taylor (1975), Taylor (1976) and Kimanani and Odhiambo (1993), and they confirmed the former results of Knipling (1955) [46] and others that used deterministic models.

As a former model, Knipling (1955, 1959) derived a simple numerical model foreshadowing most future modelling developments. The key feature of Knipling's models, and found in most of all subsequent models, is the ratio of fertile males to all males in the population. Simply modifying a geometric growth model,

$$F_{t+1} = \lambda(W_t/(S + W_t))F_t$$

where F_t and W_t is the population size of females and wild males at time t, λ is the growth rate per generation, R is the release rate of sterile individuals each generation. This yields an unstable positive equilibrium for F when $R = R*$, where $R* = F(\lambda - 1)$ denotes the critical release rate, so that if $R > R*$ then the population collapses while if $R < R*$ then the population will increase indefinitely.

The question of the competitive ability of males was modelled amongst others by Berryman (1967), Bogyo et al. (1971), Berryman et al. (1973), Ito (1977), and Barclay (1982) all showing that the critical release rate increases as the competitive ability of sterilized individuals decreases.

For a general overview of the technique, we refer the reader to [27].

1.5 The generalized contact process

In the further chapters, one constructs a contact process in random environment to lead a better understanding of this ecological phenomenon. Fix growth parameters λ_1, λ_2 and release rate r.

One introduces the contact process in dynamic random environment (CP-DRE) on the graph S with parameters set $(\lambda_1, \lambda_2, r)$ as an interacting particle system $(\xi_t, \omega_t)_{t\geqslant 0} \in (\{0,1\} \times \{0,1\})^S$ that evolves through the following dynamics. The environment part $(\omega_t)_{t\geqslant 0}$ evolves independently according to

$$0 \to 1 \text{ at rate } r, \quad 1 \to 0 \text{ at rate } 1, \tag{1.5.1}$$

while the contact process part evolves at $x \in S$ according to

$$\begin{aligned} &0 \to 1 \text{ at rate } \sum_{y:\|y-x\|=1} \Big(\lambda_1 \xi(y)(1-\omega(y)) + \lambda_2 \xi(y)\omega(y)\Big), \\ &1 \to 0 \text{ at rate } 1. \end{aligned} \tag{1.5.2}$$

As we shall see, the most interesting case corresponds to $\lambda_2 \leqslant \lambda_c < \lambda_1$, where λ_c denotes the critical value of the (basic) contact process. In words, the CP-DRE depicts a basic contact process whose growth rate is either subcritical or supercritical according to a time-evolving random environment which is parametrized by a rate r.

In our framework, one understands the environment as the space-time evolution of the sterile population released at rate r while the contact process stands for the wild population. When mixed up on a site, a competition between the two species occurs, slowing down the growth of the wild individuals to a subcritical rate λ_2, if not, the wild individuals perform a supercritical contact process. Each individual dies spontaneously at rate 1.

In a traditional overview, the contact process part describes the spread of an infection, so that the environment is thought of as being an immune response, attempting to slow down the expansion of the infection.

We also make use of a different but equivalent outlook of this process, that is, one constructs a (single) multitype contact process $(\eta_t)_{t\geqslant 0}$ on $\{0,1,2,3\}^S$, where each of these values corresponds to a possible combination of values taken by the process $(\xi_t,\omega_t)_{t\geqslant 0}$. This way, a site x of S is empty if in state 0, occupied by type-1 individuals if in state 1, by type-2 individuals if in state 2 and occupied by both types *simultaneously* if in state 3.

It is important to underline that a site is occupied by a type of individuals and not as usual, by the number of individuals present standing on. We shall therefore rather think of a multicolour system.

Biologically speaking, one interprets the type-1 individuals as being the wild individuals and the type-2 as being the sterile individuals. Sites in state 3 containing both types represent sites where competition occurs. We say that sites in state 1 or 3 constitute the *wild population*.

Furthermore, in the multitype outlook we consider two kinds of action for the type-2 individuals that are reducing the growth rate in sites in state 3. In a so-called *asymmetric* case, type-2 individuals prevent births from occurring in sites they are standing on. Call it *symmetric* otherwise. Common transition rates for both cases at site x are given by

$$\begin{array}{llll} 0\to 1 & \text{at rate } \lambda_1 n_1(x,\eta)+\lambda_2 n_3(x,\eta) & 1\to 0 & \text{at rate } 1\\ 0\to 2 & \text{at rate } r & 2\to 0 & \text{at rate } 1\\ 1\to 3 & \text{at rate } r & 3\to 1 & \text{at rate } 1\\ & & 3\to 2 & \text{at rate } 1 \end{array} \tag{1.5.3}$$

in which one adds the following transition in the symmetric case

$$2\to 3 \quad \text{at rate } \lambda_1 n_1(x,\eta)+\lambda_2 n_3(x,\eta). \tag{1.5.4}$$

As competition occurs in sites in state 3, growth rate λ_2 has to be lower than growth rate λ_1 of sites in state 1 where only type-1 individuals live. One thus makes the hypothesis :

$$\lambda_2<\lambda_1. \tag{1.5.5}$$

Here, since the presence of type-2 individuals dictate the growth rate of type-1 individuals, to even inhibit births in the asymmetric case, the type-2 individuals shape a dynamic random environment for the type-1 individuals.

Both outlooks of the process are linked by the following relations :

$$\begin{array}{lcl} \eta(x)=0 & \leftrightarrow & (1-\xi(x))(1-\omega(x))=1 \\ \eta(x)=1 & \leftrightarrow & \xi(x)(1-\omega(x))=1 \\ \eta(x)=2 & \leftrightarrow & (1-\xi(x))\omega(x)=1 \\ \eta(x)=3 & \leftrightarrow & \xi(x)\omega(x)=1 \end{array}$$

In a microscopic scale, we examine survival and extinction conditions for the population, after what, taking the hydrodynamic limit, we study the behaviour of the densities of each type of population at a macroscopic scale.

1.5.1 Phase transition in dynamic random environment

Set S as the d-dimensional integer lattice $\mathbb{Z}^d$, $d \geqslant 1$. In Chapter 2, one investigates how the release rate affects the behaviour of the process.

First, we point out general properties of the system, such as necessary and sufficient conditions for the process to be monotone, then, only sufficient conditions to be in line with the construction of the process. The tricky part to prove these conditions lies in the definition of an order on the state space $\{0,1,2,3\}^{\mathbb{Z}^d}$, since a value on a given site does not correspond to the number of particles but a type. This is the interest of the next result.

Proposition. *The symmetric multitype process is monotone, in the sense that, one can construct on a same probability space two symmetric multitype processes $(\eta_t^{(1)})_{t\geqslant 0}$ and $(\eta_t^{(2)})_{t\geqslant 0}$ with respective parameters $(\lambda_1^{(1)},\lambda_2^{(1)},r^{(1)})$ and $(\lambda_1^{(2)},\lambda_2^{(2)},r^{(2)})$, such that*

$$\eta_0^{(1)} \leqslant \eta_0^{(2)} \Longrightarrow \eta_t^{(1)} \leqslant \eta_t^{(2)} \text{ a.s. for all } t \geqslant 0 \tag{1.5.6}$$

if and only if both parameters sets satisfy

1. $\lambda_2^{(1)} \leqslant \lambda_1^{(1)}$,
2. $\lambda_2^{(2)} \leqslant \lambda_1^{(2)}$,
3. $\lambda_1^{(1)} \leqslant \lambda_1^{(2)}$,
4. $\lambda_2^{(1)} \leqslant \lambda_2^{(2)}$,
5. $r^{(1)} \geqslant r^{(2)}$
6. $\lambda_1^{(1)} \leqslant 1$,
7. $\lambda_2^{(1)} \leqslant 1$,
8. $r^{(1)} \geqslant 1$.

Essentials of SIT concern the control of the population by releasing sterile individuals, the question we address now is for which values of r does the wild population survive or die out ? For this, we prove the existence and uniqueness of a phase transition with respect to the release rate r for fixed growth rates λ_1 and λ_2. The most interesting cases are discussed in the following results :

Theorem. *Suppose $\lambda_2 \leqslant \lambda_c < \lambda_1$ fixed. Consider the symmetric multitype process. There exists a unique critical value $r_c \in (0,\infty)$ such that the wild population survives if $r < r_c$ and dies out if $r \geqslant r_c$.*

Theorem. *Suppose $\lambda_c < \lambda_1$ fixed. Consider the asymmetric multitype process. There exists a unique critical value $s_c \in (0,\infty)$ such that the wild population survives if $r < s_c$ and dies out if $r \geqslant s_c$.*

This actually confirms the former conclusions done by Knipling (1955) in a deterministic model mentioned in Section 1.4.

Proofs strongly rely on the use of graphical representations and comparison with percolation processes that introduced M. Bramson and R. Durrett [11]. Using dynamic renormalization techniques from G. Grimmett et al. [2, 35], we are in particular able to describe the behaviour of the critical process. As a consequence, this allows us to discuss the competitive ability of the sterile individuals which was biologically exhibited (as mentioned in Section 1.4) : one shows the critical value increases as the competitiveness of the sterilized population decreases or as the fitness of the wild population increases.

We end up this chapter by considering the associated mean-field equations. This shows us a dynamical system featuring the densities of each type of individuals. There, we can explicit equilibria and mainly explicit numerical bounds on the transitional phase. We shall derive a rigorous proof of the convergence of the empirical densities to these macroscopic equations in Chapters 4 and 5.

1.5.2 Survival and extinction in quenched environment

In the previous chapter, we were unable to get a hand on bounds for the critical rate. Most of the arguments made use of theory of percolation, misfit to explicit criteria for the survival and extinction events. A way to come to this end is to consider the process $(\xi_t, \omega)_{t\geqslant 0}$ by restricting the random environment to be initially fixed and setting $S = \mathbb{Z}$.

Using former results obtained by T.M. Liggett [52, 53], one obtains in Chapter 3 several survival and extinction conditions for the process. In that way, we consider two kinds of growth rates in $\mathbb{Z}$: one where the rates depend on the edges and one where the rates depend on the vertices. This yields numerical bounds on the transitional phase for the process to survive or die out.

After having investigated the behaviour of each type of individuals in a microscopic scale, we now turn into the study of the system in a macroscopic scale. When the microscopic evolution is more intricate, by a suitable scaling in time and space, we investigate the convergence of the empirical densities of each type of population.

1.5.3 Hydrodynamic limit in a bounded domain

In Chapter 4, set $S = \mathbb{T}^d$ the d-dimensional torus, and assume the microscopic dynamics is driven by the asymmetric multitype process $(\eta_t)_{t\geqslant 0}$ along with a diffusion process, modelling the migrations of the individuals. The diffusion process we consider here is a stirring process that exchanges two neighbouring occupation variables. Resulting with a reaction-diffusion process, we prove the convergence of the time-evolution of the empirical densities to the weak solution of a reaction-diffusion system.

1.5.4 Hydrodynamic limits with stochastic reservoirs or in infinite volume

One of the recurring reasons why the SIT fails, comes from an unexpected immigration in the system that prevents to maintain the pest population at a low level after regular releases. Such migrations with the external of the targeted area suggests the microscopic system is likely to be in non-equilibrium states.

In Chapter 5, one considers the microscopic time-evolution to be driven by the CP-DRE along with a rapid-stirring process. We consider a bounded cylinder connected to stochastic reservoirs at its boundaries with different densities in a stationary regime, creating and annihilating individuals. Such reservoirs create a flow through the system that put it in a nonequilibrium state, as dynamics within the bulk is no more reversible. Jointly with M. Mourragui and E. Saada, we establish the limiting equations given by a non-linear reaction-diffusion system with Dirichlet boundary conditions and a law of large numbers for the empirical currents. In a second step, we derive the hydrodynamic limit of the CP-DRE with rapid-stirring in infinite volume $\mathbb{Z}^d$.

2

Phase transition on the d-dimensional integer lattice

Contents

2.1 Introduction

The *Sterile insect technique* concerns the control of a population by releasing sterile individuals of the same species, leading to a competition with the wild individuals to the reproduction. When a match with sterile individuals occurs, offsprings reach neither the adult phase nor sexual maturity, reducing the next generation.

This chapter is an attempt to understand the behaviour of the wild population with respect to the release of the competitive sterile individuals in this model. Following issues corresponding to biology and ecology, a wide class of multi-type contact processes has emerged. Relevant questions are to identify the mechanisms involving survival, existence or coexistence of species ; such questions have been topics of works such as the grass-bushes-tree model by R. Durrett and G. Swindle [26], a 2-type contact process by C. Neuhauser [65], a 3-type model by R. Durrett and C. Neuhauser [23] for the spread of a plant disease.

The populations we consider are composed of wild males whereby sterile males are released at rate r to curb their development. We investigate the survival of the wild ones whose growth rate is time-evolving and randomly determined depending on the dynamics of the sterile individuals.

In Section 2.2, we describe the model and introduce some preliminary results about stochastic order and percolation. Then, we build graphically the particle system through Harris' graphical representation in Section 2.3. After exhibiting necessary and sufficient conditions for monotonicity properties in Section 2.4, we prove the existence and uniqueness of a phase transition with respect to the release rate in Sections 2.5 and 2.6.

2.2 Settings and results

2.2.1 The model

On the state space $\Omega = F^S$, where $F = \{0, 1, 2, 3\}$ and $S = \mathbb{Z}^d$, the *multitype contact process* is an interacting particle system $(\eta_t)_{t\geqslant 0}$ whose configuration at time t is $\eta_t \in \Omega$, that is, for all $x \in \mathbb{Z}^d$, $\eta_t(x) \in F$ represents the state of site x at time t. Two sites x and y are nearest neighbours on $\mathbb{Z}^d$ if $\|x - y\| = 1$, also written $x \sim y$, and $n_i(x, \eta_t)$ stands for the number of nearest neighbours of x in state i, $i = 1, 3$.

One understands the model as follows : at time t, a site x in $\mathbb{Z}^d$ is empty if in state 0, occupied by type-1 individuals if in state 1, by type-2 individuals if in state 2 and by both type-1 and type-2 individuals if in state 3.

Note that we only consider the type of individuals standing on each site and not their number. Moreover, we assume no limit on the number of female individuals, which is biologically a reasonable assumption (see Chapter 1).

Type-2 individuals act in two possible ways, they will reduce the growth rate of the type-1 individuals within sites in state 3. There, a competition occurs, so that the growth rate λ_2 shall be lower than the regular growth rate λ_1 in type-1 population where stand only wild individuals. Our basic assumption is thus,

$$\lambda_2 < \lambda_1. \tag{2.2.1}$$

Furthermore, in a so-called *asymmetric* case, type-2 individuals will stem births on sites they occupy.

Since we deal with the evolution of a population modelled by a particle system, we will often mingle the terms “individuals” and “particles”.

The multitype contact process. Common transitions to both cases are the following : individuals on a site in state 1 (resp. 3) gives birth to type-1 individuals at rate λ_1 (resp. λ_2) on one of its $2d$ nearest neighbour sites, if empty. A type-2 individual is dropped independently and spontaneously at rate r on any site in $\mathbb{Z}^d$. Each type dies at rate 1, deaths are mutually independent. In the so-called symmetric case, births occurs on sites in state 2 as well.

Transition rates in x for a current configuration η that are common to both cases are :

$$\begin{array}{llll} 0 \to 1 \ \text{ at rate } \lambda_1 n_1(x,\eta) + \lambda_2 n_3(x,\eta) & & 1 \to 0 \ \text{ at rate } 1 \\ 0 \to 2 \ \text{ at rate } r & & 2 \to 0 \ \text{ at rate } 1 \\ 1 \to 3 \ \text{ at rate } r & & 3 \to 1 \ \text{ at rate } 1 \\ & & 3 \to 2 \ \text{ at rate } 1 \end{array} \tag{2.2.2}$$

to which one adds the following transition in the symmetric case

$$2 \to 3 \ \text{ at rate } \lambda_1 n_1(x,\eta) + \lambda_2 n_3(x,\eta). \tag{2.2.3}$$

Therefore, the evolution of type-2 individuals occurs whatever the evolution of type-1 individuals is. Since type-2 individuals dictate the growth rate and even inhibit births in the asymmetric case, the type-2 individuals shape a *dynamic random environment* for the type-1 individuals.

In both cases, if $\eta \in \Omega$ and $x \in \mathbb{Z}^d$, denote by $\eta_x^i \in \Omega$, $\ i \in \{0,1,2,3\}$, the configuration obtained from η after a flip of x to state i :

$$\eta \longrightarrow \eta_x^i \text{ at rate } c(x,\eta,i), \text{ where } \forall u \in \mathbb{Z}^d,\ \eta_x^i(u) = \begin{cases} \eta(u) & \text{if } u \neq x \\ i & \text{if } u = x \end{cases} \tag{2.2.4}$$

Let $\mathcal{L}$ be the infinitesimal generator of $(\eta_t)_{t\geqslant 0}$, then for any cylinder function f on Ω :

$$\mathcal{L}f(\eta) = \sum_{x\in\mathbb{Z}^d}\sum_{i=0}^{3} c(x,\eta,i)\big(f(\eta_x^i) - f(\eta)\big) \tag{2.2.5}$$

with infinitesimal transition rates, common to both cases,

$$\begin{aligned} c(x,\eta,0) &= 1 \text{ if } \eta(x) \in \{1,2\} \\ c(x,\eta,1) &= \begin{cases} \lambda_1 n_1(x,\eta) + \lambda_2 n_3(x,\eta) \text{ if } \eta(x) = 0 \\ 1 \text{ if } \eta(x) = 3 \end{cases} \\ c(x,\eta,2) &= \begin{cases} r \text{ if } \eta(x) = 0 \\ 1 \text{ if } \eta(x) = 3 \end{cases} \\ c(x,\eta,3) &= \ r \text{ if } \eta(x) = 1 \end{aligned} \tag{2.2.6}$$

and add the following rate in the symmetric case :

$$c(x, \eta, 3) = \lambda_1 n_1(x, \eta) + \lambda_2 n_3(x, \eta) \text{ if } \eta(x) = 2.$$

Notice that all rates satisfy for all $i \in F$,

$$c(x, \eta, i) \geqslant 0, \ \sup_{x,\eta} c(x, \eta, i) < \infty, \tag{2.2.7}$$

$$\sup_{x\in\mathbb{Z}^d} \sum_{u\in\mathbb{Z}^d} \sup_{\eta} |c(x, \eta_u, i) - c(x, \eta, i)| < \infty. \tag{2.2.8}$$

Under these mild conditions, by Theorem 1.1.3 there exists a unique Markov process associated to the generator (2.2.5). Denote by $(\eta_t^A)_{t\geqslant 0}$ the process starting from A, i.e. such that $\eta_0 = \mathbf{1}_A$, in other words η_0 corresponds to the configuration containing sites in state 1 in A and empty otherwise. We care about the evolution of the wild population, i.e. individuals contained in sites in state 1 and 3. Define

$$H_t^A = \{x \in \mathbb{Z}^d : \eta_t^A(x) \in \{1, 3\}\}, \tag{2.2.9}$$

as the set of sites containing the wild population at time $t \geqslant 0$. Note that since $\eta_0 = \{0\}$, $H_0^{\{0\}} = \{x \in \mathbb{Z}^d : \eta_0^{\{0\}}(x) = 1\}$.

Denote by $\mathbb{P}_{\lambda_1,\lambda_2,r}$ the distribution of $(\eta_t^{\{0\}})_{t\geqslant 0}$ with parameters $(\lambda_1, \lambda_2, r)$. For fixed λ_1 and λ_2, simplify by $\mathbb{P}_r$.

Definition 2.2.1. The process $(\eta_t)_{t\geqslant 0}$ with initial configuration $\eta_0 = \mathbf{1}_{\{0\}}$, *survives* if

$$\mathbb{P}_{\lambda_1,\lambda_2,r}(\forall t \geqslant 0, \ H_t^{\{0\}} \neq \varnothing) > 0 \tag{2.2.10}$$

and *dies out* if

$$\mathbb{P}_{\lambda_1,\lambda_2,r}(\exists t \geqslant 0, \ H_t^{\{0\}} = \varnothing) = 1. \tag{2.2.11}$$

Define the *critical value* according to the parameter r by

$$r_c = r_c(\lambda_1, \lambda_2) := \inf\{r > 0 : \mathbb{P}_r(\exists t \geqslant 0, \ H_t^{\{0\}} = \varnothing) = 1\} \tag{2.2.12}$$

Indeed, the class $\{0, 2\}$ is a trap : as soon as $H_t = \varnothing$, the wild population is extinct while sterile individuals are constantly dropped along the time.

Recall λ_c stands for the critical value of the basic contact process. The purpose of this chapter is to settle the following results.

We begin by a first set of conditions for the process to survive or die out, when $\lambda_2 < \lambda_1$ are both smaller or larger than λ_c :

Proposition 2.2.1. *Suppose $\lambda_2 < \lambda_1 \leqslant \lambda_c$. For all $r \geqslant 0$, both symmetric and asymmetric multitype processes with parameters $(\lambda_1, \lambda_2, r)$ die out.*

Proposition 2.2.2. *Suppose $\lambda_c < \lambda_2 < \lambda_1$. For all $r \geqslant 0$, the symmetric multitype process with parameters $(\lambda_1, \lambda_2, r)$ survives.*

The most interesting cases are given by

Theorem 2.2.1. *Suppose $\lambda_2 < \lambda_c < \lambda_1$. Consider the symmetric multitype process. There exists a unique critical value $r_c \in (0, \infty)$ such that if $r < r_c$, then the process survives and if $r > r_c$, then the process dies out.*

Theorem 2.2.2. *Suppose $\lambda_c < \lambda_1$ and $\lambda_2 < \lambda_1$. Consider the asymmetric multitype process. There exists a unique critical value $s_c \in (0, \infty)$ such that if $r < s_c$, then the process survives and if $r > s_c$, then the process dies out.*

In both cases, one has

Theorem 2.2.3. *The critical multitype process dies out.*

The next two subsections are setting preliminaries to prove these results.

2.2.2 Necessary and sufficient conditions for attractiveness

We saw in Chapter 1 the stochastic order between two processes is related to the total order defined on the set of values taken by both processes, here on $F = \{0, 1, 2, 3\}$. In a biological context, setting an order between types of individuals does not make any sense, but mathematically it allows us to construct a monotone model and to compare different dynamics as well. This is the purpose of Section 2.4, using Theorem 2.2.4 below. Elements of F can be understood as species of respective types A, B, C and D. A process can be made attractive by reordering its space of values. Subsequently, denote by A the state 2, by B the state 0, by C the state 3 and by D the state 1, ordered by

$$A < B = A + 1 < C = B + 1 < D = C + 1. \tag{2.2.13}$$

Extending conditions obtained by T. Gobron and E. Saada [31] for conservative particle systems, D. Borrello [10] has settled necessary and sufficient conditions to non conservative dynamics to determine stochastic order between two processes. Particularly, [10, section 2.2.2] deals with multitype contact processes corresponding to our framework. We will see that this order is actually the only possible one that preserves the stochastic order.

Let $x, y \in \mathbb{Z}^d$ be two neighbouring sites and $\alpha, \beta \in F$, rewrite the transition rates of $(\eta_t)_{t \geqslant 0}$ with notations of [10], for $k \in \{1, 2\}$, as

- $R^{0,k}_{\alpha,\beta}$ the growth rate of a type-1 individual in y such that $\eta(y) = \beta$, depending only on the value of $\eta(x) = \alpha$. The state in y flips from β to $\beta + k$.
- P^k_β the jump rate of a site from state $\eta(y) = \beta$ to state $\beta + k$, depending only on the value of $\eta(y)$.

- P_α^{-k} the jump rate of a site from state $\eta(x) = \alpha$ to state $\alpha - k$ for $k \leqslant \alpha$, depending only on the value of $\eta(x)$.

Next, define

$$\Pi^{0,k}_{\alpha,\beta} := R^{0,k}_{\alpha,\beta} + P^k_\beta \text{ and } \Pi^{-k,0}_{\alpha,\beta} := P^{-k}_\alpha. \tag{2.2.14}$$

Theorem 2.2.4. *[10, Theorem 2.4] For all $(\alpha, \beta) \in F^2$, $(\gamma, \delta) \in F^2$ such that $(\alpha, \beta) \leqslant (\gamma, \delta)$ (coordinate-wise, in the sense that $\alpha \leqslant \gamma$ and $\beta \leqslant \delta$), $h_1 \geqslant 0$, $j_1 \geqslant 0$, an interacting particle systems $(A_t)_{t\geqslant 0}$ with transition rates $(R^{0,k}_{\alpha,\beta}, P^{+k}_\beta, P^{-k}_\alpha)$ is stochastically larger than an interacting particle system $(B_t)_{t\geqslant 0}$ with transition rates $(\widetilde{R}^{0,k}_{\alpha,\beta}, \widetilde{P}^{+k}_\beta, \widetilde{P}^{-k}_\alpha)$ if and only if*

$$i) \sum_{k>\delta-\beta+j_1} \widetilde{\Pi}^{0,k}_{\alpha,\beta} \leqslant \sum_{l>j_1} \Pi^{0,l}_{\gamma,\delta} \quad \textit{and} \quad ii) \sum_{k>h_1} \widetilde{\Pi}^{-k,0}_{\alpha,\beta} \geqslant \sum_{l>\gamma-\alpha+h_1} \Pi^{-l,0}_{\gamma,\delta} \tag{2.2.15}$$

One has for the asymmetric multitype process $(\eta_t)_{t\geqslant 0}$, with the order (2.2.13), the following rates.

$$\begin{array}{c} R^{0,2}_{D,B} = \lambda_1, \quad R^{0,2}_{C,B} = \lambda_2, \\ P^1_A = P^1_C = 1, \\ P^{-1}_B = P^{-1}_D = r, \\ P^{-2}_C = P^{-2}_D = 1, \end{array} \tag{2.2.16}$$

to which, one adds the following rates if we consider the symmetric multitype process.

$$R^{0,2}_{D,A} = \lambda_1, \quad R^{0,2}_{C,A} = \lambda_2. \tag{2.2.17}$$

Similarly, for a basic contact process with growth rate λ_1 on $\{0,1\}^{\mathbb{Z}^d}$, one has

$$\widetilde{R}^{0,2}_{D,B} = \lambda_1, \ \widetilde{P}^{-2}_D = 1. \tag{2.2.18}$$

It will be also useful to consider a basic contact process with growth rate λ_2, defined on $\{2,3\}^{\mathbb{Z}^d}$, with rates

$$\widetilde{R}^{0,2}_{C,A} = \lambda_2, \ \widetilde{P}^{-2}_C = 1. \tag{2.2.19}$$

2.2.3 Oriented percolation

In the following, we give a brief description presented by R. Durrett [19] about oriented percolation and the comparison theorem, and their correspondence with interacting particle systems. The first application of this technique was done by M. Bramson and R. Durrett [11] for spin systems.

Construction. Here is a description of an *oriented (site) percolation process* with parameter p. Consider the bi-dimensional even lattice

$$\mathcal{L} = \{(x,n) \in \mathbb{Z}^2 : x + n \text{ is even}, n \geqslant 0\}.$$

From $\mathcal{L}$, construct an oriented graph by drawing successively an oriented bond from (x,n) to $(x+1,n+1)$ and one from (x,n) to $(x-1,n+1)$. Let $\{\omega(x,n),(x,n)\in\mathcal{L}\}$ be random variables taking their values in $\{0,1\}$ that indicate whether a site of $\mathcal{L}$ is open (1) or closed (0). We define their distribution in what follows.

There is an (oriented) open path from (x,n) to (y,m), denoted by $(x,m)\to(y,n)$, if there exists a sequence of points $x=x_n,...,x_m=y$ such that $(x_k,k)\in\mathcal{L}$, $|x_k-x_{k+1}|=1$ for $n\leqslant k\leqslant m-1$ and $\omega(x_k,k)=1$ for $n\leqslant k\leqslant m$. Since in our further setup, our constructions will set dependencies between the $\omega(x,n)$'s, we say that the $\omega(x,n)$'s are *M-dependent with density at least* $1-\gamma$, for positive M and γ, if whenever $(x_k,n_k)_{1\leqslant k\leqslant I}$ is a finite sequence such that $\|(x_i,n_i)-(x_j,n_j)\|_\infty>M$ for $i\neq j$ then

$$P(\omega(x_i,n_i)=0 \text{ for } 1\leqslant i\leqslant |I|)\leqslant\gamma^I.$$

Oriented percolation is understood as a mimic of the crossing of fluids through some porous materials along a given direction, as a flow of water in a porous rock. Therefore, open sites are understood as air spaces the fluid can reach and turning them into wet sites if reached. Varying the microscopic porosity of the spaces (given by the distribution of ω), percolation processes exhibit a macroscopic phase transition from a permeable percolating regime to an impermeable non-percolating regime.

Given an initial condition $W_0\subset 2\mathbb{Z}=\{x\in\mathbb{Z}:(x,0)\in\mathcal{L}\}$, we introduce the process of wet sites at time $n\geqslant 0$ by

$$W_n:=\{y:(x,0)\to(y,n) \text{ for some } x\in W_0\}$$

Let W_n^0 be the process starting from $W_0^0=\{0\}$ and define

$$C_0:=\{(y,n):(0,0)\to(y,n)\}$$

as the set of points reached by the origin $(0,0)$ through an oriented open path. It is also called the *connected open component or cluster from the origin*. When the latter is infinite, that is, $\{|C_0|=\infty\}$, we say that *percolation occurs*.

A natural question is whether percolation occurs or not. The following result shows that if the density of open sites is high enough then percolation occurs with positive probability :

Theorem 2.2.5 (R. Durrett [18])**.** *If $\gamma\leqslant 6^{-4(2M+1)^2}$, then*

$$P(|C_0|<\infty)\leqslant 1/20$$

Percolation processes that will arise are M-dependent but since most of the literature concerns percolation with independent random variables, next theorem tells us how a M-dependent process stochastically dominates the measure of a 0-dependent percolation. Let π_p be the product measure of an independent percolation process with density p, i.e. with cylinder probabilities

$$\pi_p(\omega:\omega(x,n)=1\ \forall(x,n)\in G;\ \omega(x,n)=0\ \forall(x,n)\in H)=p^{|G|}(1-p)^{|H|}.$$

where G,H are finite subsets of $\mathcal{L}$. We have in our setup,

Theorem 2.2.6 (Liggett, Schonmann and Stacey [59]). *Let μ be a 1-dependent Bernoulli distribution. If*

$$\mu(\omega(x,n)=1) \geqslant 1-(1-\sqrt{p})^2 \ a.s.$$

for all $(x,n) \in \mathcal{L}$ with $p \geqslant 1/4$, then $\mu \geqslant \pi_p$.

So far, the link between an interacting particle system and a percolation process is still missing, this is the point of what follows.

Comparison theorem. The next result gives general conditions guaranteeing a process to dominate an oriented percolation.

(H) Comparison Assumptions. Let be $(\xi_t)_{t\geqslant 0}$ a translation invariant finite range process such that $\xi_t \in F^{\mathbb{Z}^d}$, constructed from a graphical representation. Given positive integers L, T, k_0 and j_0, define for $(m,n) \in \mathcal{L}$, space-time regions

$$\mathcal{R}_{m,n} = (2mLe_1, nT) + \big([-k_0L, k_0L]^d \times [0, j_0T]\big) \tag{2.2.20}$$

where $(e_1, ..., e_d)$ stands for the canonical basis in $\mathbb{R}^d$. Let $M := \max(k_0, j_0)$, the regions $\mathcal{R}_{m,n}$ and $\mathcal{R}_{m',n'}$ are disjoint if $\|(m,n)-(m',n')\|_\infty > M$.

Let H be collection of configurations determined by the values of ξ in $[-L,L]^d$. We declare $(m,n) \in \mathcal{L}$ to be *wet* if $\xi_{nT} \in \tau_{2mLe_1}H$, where τ_{Le_1} stands for the translation by L in the direction e_1.

Suppose, for all $(m,n) \in \mathcal{L}$, there exists a *good event* $G_{m,n}$ depending only on the graphical representation of the particle system in $\mathcal{R}_{m,n}$ such that $P(G_{m,n}) \geqslant 1-\theta$ $(\theta > 0)$ and so that if (m,n) is wet, then on $G_{m,n}$, $(m+1,n+1)$ and $(m-1,n+1)$ do as well, that is,

$$\xi_{(n+1)T} \in \tau_{2(m-1)Le_1}H \text{ and } \xi_{(n+1)T} \in \tau_{2(m+1)Le_1}H.$$

Let $X_n = \{m : (m,n) \in \mathcal{L}, \xi_{nT} \in \tau_{2mLe_1}H\}$ be the set of wet sites at time t. Then,

Theorem 2.2.7. *[19, Theorem 4.3] If the comparison assumptions (**H**) hold, then one can define random variables $\omega(x,n)$ so that for all $n \geqslant 0$, X_n dominates an M-dependent oriented percolation with initial configuration $W_0 = X_0$ and density at least $1-\gamma$, that is,*

$$W_n \subset X_n \text{ for all } n.$$

2.3 Graphical construction

In parallel to the analytical construction provided by the Hille-Yosida theorem 1.1.2, the multitype contact process can be constructed from a collection of independent Poisson processes [38]. Think of the diagram $\mathbb{Z}^d \times \mathbb{R}_+$. For each $x \in \mathbb{Z}^d$, consider the arrival times of mutually independent families of Poisson processes : $\{A_n^x : n \geqslant 1\}$ with rate

r, $\{D_n^{1,x} : n \geqslant 1\}$ and $\{D_n^{2,x} : n \geqslant 1\}$ with rate 1 and for any y such that $y \sim x$, $\{T_n^{x,y} : n \geqslant 1\}$ with rate λ_1. Let $\{U_n^x : n \geqslant 1\}$ be independent uniform random variables on $(0, 1)$, independent of the Poisson processes.

At space-time point (x, A_n^x), put a "$\bullet$" to indicate, if x is occupied by type-1 individuals (resp. empty), that it turns into state 3 (resp. state 2) which corresponds to transitions $0 \to 2$ and $1 \to 3$. At $(x, D_n^{1,x})$ (resp. at $(x, D_n^{2,x})$), put an "X" (resp."$\circ$") to indicate at x, that a death of type-1 occurs corresponding to transitions $3 \to 2$ and $1 \to 0$ (resp. of type-2, corresponding to transitions $3 \to 1$ and $2 \to 0$). At times $T_n^{x,y}$, draw an arrow from x to y and two kinds of actions occur following the occupation at x : if x is occupied by type-1 individuals, the arrow indicates a birth in y of a type-1 individual if y is empty or in state 2, corresponding to transitions $0 \to 1$, and $2 \to 3$ for the symmetric case ; if x is occupied by type-3 individuals giving birth at rate $\lambda_2 < \lambda_1$, check at $(x, T_n^{x,y})$ if $U_n^x < \lambda_2/\lambda_1$ to indicate, if success, that the arrow is effective so that a birth in y of a type-1 individual occurs if y is empty, or in state 2 for the symmetric case. In the asymmetric case, births occur only if y is not in state 2.

See Figure 2.1 for an example of the time-evolution of both processes starting from an identical initial configuration.

For $s \leqslant t$, there exists an *active path from* (x, s) *to* (y, t) in $\mathbb{Z}^d \times \mathbb{R}_+$ is there exists a sequence of times $s = s_0 < s_1 < ... < s_{n-1} < s_n = t$ and a sequence of corresponding spatial locations $x = x_0, x_1, ..., x_n = y$ such that :

i. for $i = 1, ..., n-1$, vertical segments $\{x_i\} \times (s_i, s_{i+1})$ do not contain any X's.

ii. for $i = 1, ..., n$, there is an arrow from x_{i-1} to x_i at times s_i and if $x_{i-1} \times s_i$ is lastly preceded by a " $\bullet$" this arrow exists only if $U_{s_i}^{x_{i-1}} < \lambda_2/\lambda_1$.

and in the asymmetric case, substitute ii. by

ii'. for $i = 1, ..., n$, there is an arrow from x_{i-1} to x_i at times s_i while $\{x_i\} \times s_i$ is not lastly preceded by a "$\bullet$", while if $x_{i-1} \times s_i$ is lastly preceded by a "$\bullet$" this arrow is effective if $U_{s_i}^{x_{i-1}} < \lambda_2/\lambda_1$.

Consider the process $(A_t^A)_{t \geqslant 0}$, the set of sites at time t reached by active paths starting from an initial configuration $A_0 = A$, containing sites in state 1 in A and 0 otherwise :

$$A_t^A = \{y \in \mathbb{Z} : \exists x \in A \text{ such that } (x, 0) \to (y, t)\}$$

Then $A_t^A = H_t^A$, with H_t^A defined in (2.2.9) so that A_t^A represents the wild population at time t starting from an initial configuration A of type-1 individuals.

From the graphical representation, the particle system $(A_t^A)_{t \geqslant 0}$ is additive [32, Chapter II] : for any initial configuration B such that $A \subset B$, then

$$A_t^A \subset A_t^B.$$

On Figure 2.1, $A_t^{\{0\}} = \{-1, 0, 1\}$ for the asymmetric case. This graphical representation allows us to couple multitype contact processes starting from different initial

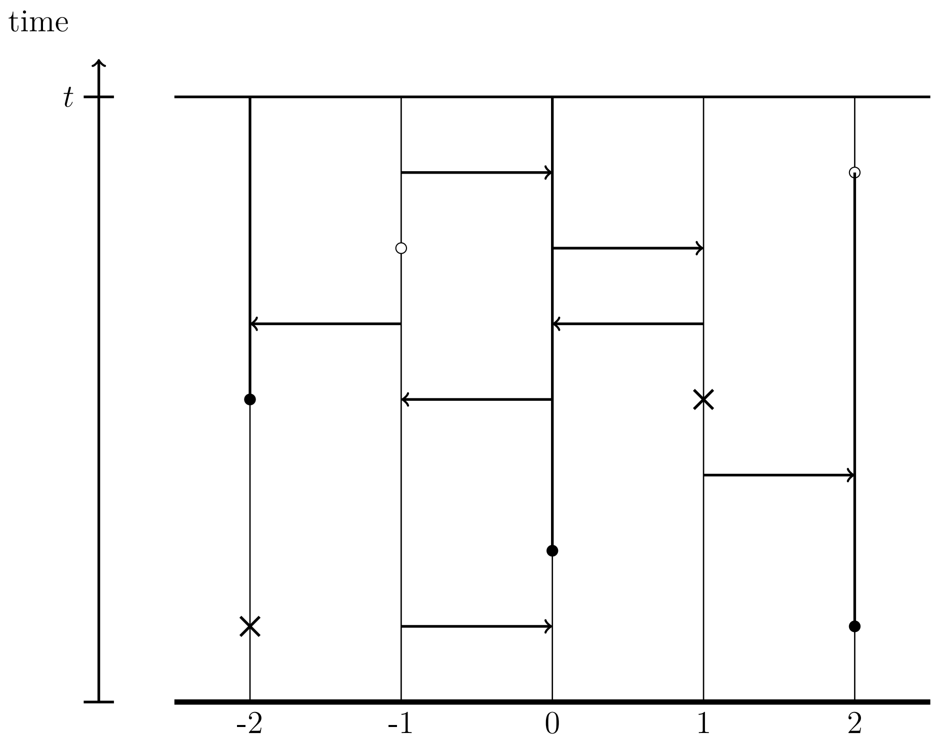

FIGURE 2.1: Graphical representation in the space-time picture $\mathbb{Z} \times \mathbb{R}_+$. Starting from $\eta_0 = \mathbf{1}_{\{0\}}$, following the arrows, if $U_1^0 < \frac{\lambda_2}{\lambda_1}$ and $U_2^0 < \frac{\lambda_2}{\lambda_1}$, the wild population occupies at time t the set $H_t = \{-1, 0, 1\}$ in the asymmetric case and the set $H_t = \{-2, -1, 0, 1\}$ in the symmetric case.

configurations by imposing the evolution to obey to the same Poisson processes. Other kinds of couplings would be possible through the analytical construction of the process as we will see later. By way of illustration, $A_t^{\{1\}} = \varnothing$ and $A_t^{\{0,1\}} = \{-1, 0, 1\}$ in the asymmetric case, $A_t^{\{1\}} = \{2\}$ and $A_t^{\{0,1\}} = \{-2, -1, 0, 1, 2\}$ in the symmetric case. More generally, graphical representations allow to couple processes with different dynamics as well, we investigate this question furthermore thereafter.

2.4 Attractiveness and stochastic order

Recall $(\eta_t)_{t\geqslant 0}$ denotes the multitype contact process with parameters $(\lambda_1, \lambda_2, r)$ and $(\xi_t)_{t\geqslant 0}$ denotes the basic contact process with growth rate λ_1. Most of the proofs below rely on the construction of a markovian coupled process.

We defined a partial order on $F^{\mathbb{Z}^d}$ between two configurations $\eta^{(1)}$ and $\eta^{(2)}$ by (1.1.3) and (2.2.13). Here we shall settle necessary and sufficient conditions, then only sufficient, to obtain several properties of stochastic order with which we will work. We begin with the symmetric multitype contact process since it contains the transitions of the asymmetric one and of the basic contact process.

Proposition 2.4.1. *The symmetric multitype process is monotone, in the sense that, one can construct on a same probability space two symmetric multitype processes $(\eta_t^{(1)})_{t\geqslant 0}$ and $(\eta_t^{(2)})_{t\geqslant 0}$ with respective parameters $(\lambda_1^{(1)}, \lambda_2^{(1)}, r^{(1)})$ and $(\lambda_1^{(2)}, \lambda_2^{(2)}, r^{(2)})$, such that*

$$\eta_0^{(1)} \leqslant \eta_0^{(2)} \Longrightarrow \eta_t^{(1)} \leqslant \eta_t^{(2)} \text{ a.s. for all } t \geqslant 0 \tag{2.4.1}$$

if and only if all parameters satisfy

1. $\lambda_2^{(1)} \leqslant \lambda_1^{(1)}$,	*3.* $\lambda_1^{(1)} \leqslant \lambda_1^{(2)}$,	*5.* $r^{(1)} \geqslant r^{(2)}$	*7.* $\lambda_2^{(1)} \leqslant 1$,
2. $\lambda_2^{(2)} \leqslant \lambda_1^{(2)}$,	*4.* $\lambda_2^{(1)} \leqslant \lambda_2^{(2)}$,	*6.* $\lambda_1^{(1)} \leqslant 1$,	*8.* $r^{(1)} \geqslant 1$.

Remark conditions 1. and 2. are the assumptions made from the construction of the process, see (2.2.1).

Proof of Proposition 2.4.1. Let $(\eta_t^{(1)})_{t\geqslant 0}$ and $(\eta_t^{(2)})_{t\geqslant 0}$ be two symmetric processes with parameters $(\lambda_1^{(1)}, \lambda_2^{(1)}, r^{(1)})$ and $(\lambda_1^{(2)}, \lambda_2^{(2)}, r^{(2)})$ respectively. Apply Theorem 2.2.4 with $j_1, h_1 \in \{0, 1\}$ (one can check they are the only non trivial possible values). Necessary and sufficient conditions on the rates for $(\eta_t^{(2)})_{t\geqslant 0}$ to be stochastically larger than $(\eta_t^{(1)})_{t\geqslant 0}$ are given by relations (2.2.15) with $(\alpha, \beta) \leqslant (\gamma, \delta)$, that is,

$$\sum_{k>\delta-\beta+j_1} \Pi_{\alpha,\beta,(1)}^{0,k,(1)} \leqslant \sum_{l>j_1} \Pi_{\gamma,\delta}^{0,l,(2)} \quad \text{and} \quad \sum_{k>h_1} \Pi_{\alpha,\beta}^{-k,0,(1)} \geqslant \sum_{l>\gamma-\alpha+h_1} \Pi_{\gamma,\delta}^{-l,0,(2)}$$

with the rates previously defined by (2.2.16)-(2.2.17). One then has

$$
\begin{aligned}
&\mathbf{1}\{j_1=0\}\mathbf{1}\{k=2\}\mathbf{1}\{\delta-\beta=1\}\Big(\mathbf{1}\{\delta=C,\beta=B\}\big(R^{0,2,(1)}_{D,B}\mathbf{1}\{\alpha=D\}+R^{0,2,(1)}_{C,B}\mathbf{1}\{\alpha=C\}\big)\\
&\qquad+\mathbf{1}\{\delta=B,\beta=A\}\big(R^{0,2,(1)}_{D,A}\mathbf{1}\{\alpha=D\}+R^{0,2,(1)}_{C,A}\mathbf{1}\{\alpha=C\}\big)\Big)\\
&+\mathbf{1}\{j_1=0\}\mathbf{1}\{k=2\}\mathbf{1}\{\delta-\beta=0\}\Big(\mathbf{1}\{\delta=\beta=B\}\big(R^{0,2,(1)}_{D,B}\mathbf{1}\{\alpha=D\}+R^{0,2,(1)}_{C,B}\mathbf{1}\{\alpha=C\}\big)\\
&\qquad+\mathbf{1}\{\delta=\beta=A\}\big(R^{0,2,(1)}_{D,A}\mathbf{1}\{\alpha=D\}+R^{0,2,(1)}_{C,A}\mathbf{1}\{\alpha=C\}\big)\Big)\\
&+\mathbf{1}\{j_1=0\}\mathbf{1}\{k=1\}\mathbf{1}\{\delta-\beta=0\}\Big(\mathbf{1}\{\delta=\beta=C\}P^{1,(1)}_{C}+\mathbf{1}\{\delta=\beta=A\}P^{1,(1)}_{A}\Big)\\
&+\mathbf{1}\{j_1=1\}\mathbf{1}\{k=2\}\mathbf{1}\{\delta-\beta=0\}\\
&\qquad\Big(\mathbf{1}\{\delta=\beta=B\}\big(R^{0,2,(1)}_{D,B}\mathbf{1}\{\alpha=D\}+R^{0,2,(1)}_{C,B}\mathbf{1}\{\alpha=C\}\big)\\
&\qquad+\mathbf{1}\{\delta=\beta=A\}\big(R^{0,2,(1)}_{D,A}\mathbf{1}\{\alpha=D\}+R^{0,2,(1)}_{C,A}\mathbf{1}\{\alpha=C\}\big)\Big)\\
&\leqslant\mathbf{1}\{j_1=0\}\mathbf{1}\{l=2\}\Big(\mathbf{1}\{\delta=B\}\big(R^{0,2,(2)}_{D,B}\mathbf{1}\{\gamma=D\}+R^{0,2,(2)}_{C,B}\mathbf{1}\{\gamma=C\}\big)\\
&\qquad+\mathbf{1}\{\delta=A\}\big(R^{0,2,(2)}_{D,A}\mathbf{1}\{\gamma=D\}+R^{0,2,(2)}_{C,A}\mathbf{1}\{\gamma=C\}\big)\Big)\\
&+\mathbf{1}\{j_1=0\}\mathbf{1}\{l=1\}\Big(\mathbf{1}\{\delta=C\}P^{1,(2)}_{C}+\mathbf{1}\{\delta=A\}P^{1,(2)}_{A}\Big)\\
&+\mathbf{1}\{j_1=1\}\mathbf{1}\{l=2\}\Big(\mathbf{1}\{\delta=B\}\big(R^{0,2,(2)}_{D,B}\mathbf{1}\{\gamma=D\}+R^{0,2,(2)}_{C,B}\mathbf{1}\{\gamma=C\}\big)\\
&\qquad+\mathbf{1}\{\delta=A\}\big(R^{0,2,(2)}_{D,A}\mathbf{1}\{\gamma=D\}+R^{0,2,(2)}_{C,A}\mathbf{1}\{\gamma=C\}\big)\Big)
\end{aligned}
\tag{2.4.2}
$$

and

$$
\begin{aligned}
&\mathbf{1}\{h_1=0\}\mathbf{1}\{k=1\}\Big(\mathbf{1}\{\alpha=D\}P^{-1,(1)}_{D}+\mathbf{1}\{\alpha=B\}P^{-1,(1)}_{B}\Big)\\
&+\mathbf{1}\{h_1=0\}\mathbf{1}\{k=2\}\Big(\mathbf{1}\{\alpha=D\}P^{-2,(1)}_{D}+\mathbf{1}\{\alpha=C\}P^{-2,(1)}_{C}\Big)\\
&+\mathbf{1}\{h_1=1\}\mathbf{1}\{k=2\}\Big(\mathbf{1}\{\alpha=D\}P^{-2,(1)}_{D}+\mathbf{1}\{\alpha=C\}P^{-2,(1)}_{C}\Big)\\
&\geqslant\mathbf{1}\{h_1=0\}\mathbf{1}\{l=2\}\mathbf{1}\{\gamma-\alpha=1\}\Big(\mathbf{1}\{\gamma=D,\alpha=C\}P^{-2,(2)}_{D}+\mathbf{1}\{\gamma=C,\alpha=B\}P^{-2,(2)}_{C}\big)\\
&+\mathbf{1}\{h_1=0\}\mathbf{1}\{l=2\}\mathbf{1}\{\gamma-\alpha=0\}\Big(\mathbf{1}\{\gamma=\alpha=D\}P^{-2,(2)}_{D}+\mathbf{1}\{\gamma=\alpha=C\}P^{-2,(2)}_{C}\Big)\\
&+\mathbf{1}\{h_1=0\}\mathbf{1}\{l=1\}\mathbf{1}\{\gamma-\alpha=0\}\Big(\mathbf{1}\{\gamma=\alpha=D\}P^{-1,(2)}_{D}+\mathbf{1}\{\gamma=\alpha=C\}P^{-1,(2)}_{C}\Big)\\
&\mathbf{1}\{h_1=1\}\mathbf{1}\{l=2\}\mathbf{1}\{\gamma-\alpha=0\}\Big(\mathbf{1}\{\gamma=\alpha=D\}P^{-2,(2)}_{D}+\mathbf{1}\{\alpha=\gamma=C\}P^{-2,(2)}_{C}\Big)
\end{aligned}
\quad . \tag{2.4.3}
$$

These inequalities can also be explicitly rewritten as

$$
\begin{aligned}
&\mathbf{1}\{j_1=0\}\mathbf{1}\{k=2\}\big(\mathbf{1}\{\beta=A,\delta=B\}+\mathbf{1}\{\beta=B,\delta=C\}\big)\big(\lambda^{(1)}_1\mathbf{1}\{\alpha=D\}+\lambda^{(1)}_2\mathbf{1}\{\alpha=C\}\big)\\
&+\mathbf{1}\{j_1=0\}\mathbf{1}\{k=2\}\big(\mathbf{1}\{\beta=\delta=B\}+\mathbf{1}\{\beta=\delta=A\}\big)\big(\lambda^{(1)}_1\mathbf{1}\{\alpha=D\}+\lambda^{(1)}_2\mathbf{1}\{\alpha=C\}\big)\\
&+\mathbf{1}\{j_1=0\}\mathbf{1}\{k=1\}\Big(\mathbf{1}\{\beta=\delta=A\}+\mathbf{1}\{\beta=\delta=C\}\Big)\\
&+\mathbf{1}\{j_1=1\}\mathbf{1}\{k=2\}\big(\mathbf{1}\{\beta=\delta=B\}+\mathbf{1}\{\beta=\delta=A\}\big)\big(\lambda^{(1)}_1\mathbf{1}\{\alpha=D\}+\lambda^{(1)}_2\mathbf{1}\{\alpha=C\}\big)\\
&\leqslant\mathbf{1}\{j_1=0\}\Big(\mathbf{1}\{l=2\}\big(\mathbf{1}\{\delta=B\}+\mathbf{1}\{\delta=A\}\big)\big(\lambda^{(2)}_1\mathbf{1}\{\gamma=D\}+\lambda^{(2)}_2\mathbf{1}\{\gamma=C\}\big)\Big)\\
&+\mathbf{1}\{j_1=0\}\mathbf{1}\{l=1\}\big(\mathbf{1}\{\delta=A\}+\mathbf{1}\{\delta=C\}\big)\Big)\\
&+\mathbf{1}\{j_1=1\}\mathbf{1}\{l=2\}\big(\mathbf{1}\{\delta=B\}+\mathbf{1}\{\delta=A\}\big)\big(\lambda^{(2)}_1\mathbf{1}\{\gamma=D\}+\lambda^{(2)}_2\mathbf{1}\{\gamma=C\}\big)
\end{aligned}
\tag{2.4.4}
$$

and

$$\begin{aligned}
&\mathbf{1}\{h_1=0\}\big(\mathbf{1}\{k=1\}r^{(1)}(\mathbf{1}\{\alpha=B\}+\mathbf{1}\{\alpha=D\}\big)\\
&\quad+\mathbf{1}\{h_1=0\}\mathbf{1}\{k=2\}\big(\mathbf{1}\{\alpha=C\}+\mathbf{1}\{\alpha=D\}\big)\\
&\qquad+\mathbf{1}\{h_1=1\}\mathbf{1}\{k=2\}(\mathbf{1}\{\alpha=C\}+\mathbf{1}\{\alpha=D\})\\
&\geqslant \mathbf{1}\{h_1=0\}\mathbf{1}\{l=2\}\mathbf{1}\{\gamma=1+\alpha\}\Big(\mathbf{1}\{\gamma=C\}+\mathbf{1}\{\gamma=D\}\Big)\\
&\quad+\mathbf{1}\{h_1=0\}\mathbf{1}\{l=2\}\mathbf{1}\{\gamma=\alpha\}\big(\mathbf{1}\{\gamma=C\}+\mathbf{1}\{\gamma=D\}\big)\\
&\quad+\mathbf{1}\{h_1=0\}\mathbf{1}\{l=1\}\mathbf{1}\{\gamma=\alpha\}r^{(2)}(\mathbf{1}\{\gamma=B\}+\mathbf{1}\{\gamma=D\})\\
&\qquad+\mathbf{1}\{h_1=1\}\mathbf{1}\{\gamma=\alpha\}\mathbf{1}\{l=2\}(\mathbf{1}\{\gamma=C\}+\mathbf{1}\{\gamma=D\}).
\end{aligned}\tag{2.4.5}$$

All different possible scenarios provide the following necessary conditions :

(I) $j_1\in\{0,1\}$, $\delta=\beta\in\{A,B\}$ in (2.4.4) give

 (i) $\alpha=C,\gamma=D:\lambda_2^{(1)}\leqslant\lambda_1^{(2)}$. This is a consequence of conditions 1. and 3. or 2. and 4.

 (ii) $\alpha=\gamma=C:\lambda_2^{(1)}\leqslant\lambda_2^{(2)}$ stated by condition 4.

 (iii) $\alpha=\gamma=D:\lambda_1^{(1)}\leqslant\lambda_1^{(2)}$ stated by condition 3.

(II) $j_1=0$, $\beta=B$, $\delta=1+\beta=C$ in (2.4.4) give

 (i) $\alpha=D:\lambda_1^{(1)}\leqslant 1$ stated by condition 6.

 (ii) $\alpha=C:\lambda_2^{(1)}\leqslant 1$ stated by condition 7.

(III) $h_1=0$, $\gamma=\alpha\in\{B,D\}$ in (2.4.5) give $r^{(1)}\geqslant r^{(2)}$ stated by condition 5.

(IV) $h_1=0$,$\alpha=B,\gamma=1+\alpha=C$ in (2.4.5) give $r^{(1)}\geqslant 1$ stated by condition 8.

while in other scenarios, one retrieves redundantly the above conditions or tautological inequalities such as “$1\geqslant 0$". Finally, one obtained the necessary conditions stated from 1. to 8.

Now, we construct a coupled process $(\eta_t^{(1)},\eta_t^{(2)})_{t\geqslant 0}$ on $\Omega\times\Omega$ such that $\eta_0^{(1)}\leqslant\eta_0^{(2)}$. According to the given order (2.2.13) on F, as $\eta_0^{(1)}\leqslant\eta_0^{(2)}$:

$$n_1(x,\eta_0^{(1)})+n_3(x,\eta_0^{(1)})\leqslant n_1(x,\eta_0^{(2)})+n_3(x,\eta_0^{(2)}).$$

We saw that it is possible to construct the coupled process either through generators or through a graphical representation, via a collection of independent Poisson processes whose rates are given by the parameters of the processes. The coupling of two processes on a graphical construction is provided by coupling the Poisson processes related to births and releases.

In what follows, we construct the coupling through generators. The three following tables depict the infinitesimal transitions of the coupled process.

transition		rate
$(0,0) \longrightarrow$	$(1,1)$	$\lambda_1^{(1)} n_1(x,\eta^{(1)}) + \lambda_2^{(1)} n_3(x,\eta^{(1)})$
	$(0,1)$	$\lambda_1^{(2)} n_1(x,\eta^{(2)}) - \lambda_1^{(1)} n_1(x,\eta^{(1)}) + \lambda_2^{(2)} n_3(x,\eta^{(2)}) - \lambda_2^{(1)} n_3(x,\eta^{(1)})$
	$(2,2)$	$r^{(2)}$
	$(2,0)$	$r^{(1)} - r^{(2)}$
$(1,1) \longrightarrow$	$(0,0)$	1
	$(3,3)$	$r^{(2)}$
	$(3,1)$	$r^{(1)} - r^{(2)}$
$(2,2) \longrightarrow$	$(3,3)$	$\lambda_1^{(1)} n_1(x,\eta^{(1)}) + \lambda_2^{(1)} n_3(x,\eta^{(1)})$
	$(2,3)$	$\lambda_1^{(2)} n_1(x,\eta^{(2)}) - \lambda_1^{(1)} n_1(x,\eta^{(1)}) + \lambda_2^{(2)} n_3(x,\eta^{(2)}) - \lambda_2^{(1)} n_3(x,\eta^{(1)})$
	$(0,0)$	1
$(3,3) \longrightarrow$	$(1,1)$	1
	$(2,2)$	1
$(2,0) \longrightarrow$	$(3,1)$	$\lambda_1^{(1)} n_1(x,\eta^{(1)}) + \lambda_2^{(1)} n_3(x,\eta^{(1)})$
	$(2,1)$	$\lambda_1^{(2)} n_1(x,\eta^{(2)}) - \lambda_1^{(1)} n_1(x,\eta^{(1)}) + \lambda_2^{(2)} n_3(x,\eta^{(2)}) - \lambda_2^{(1)} n_3(x,\eta^{(1)})$
	$(0,0)$	1
	$(2,2)$	$r^{(2)}$
$(2,3) \longrightarrow$	$(3,3)$	$\lambda_1^{(1)} n_1(x,\eta^{(1)}) + \lambda_2^{(1)} n_3(x,\eta^{(1)})$
	$(0,1)$	1
	$(2,2)$	1
$(2,1) \longrightarrow$	$(3,1)$	$\lambda_1^{(1)} n_1(x,\eta^{(1)}) + \lambda_2^{(1)} n_3(x,\eta^{(1)})$
	$(2,0)$	1
	$(0,1)$	1
	$(2,3)$	$r^{(2)}$
$(3,1) \longrightarrow$	$(2,0)$	1
	$(1,1)$	1
	$(3,3)$	$r^{(2)}$
$(0,1) \longrightarrow$	$(2,3)$	$r^{(2)}$
	$(1,1)$	$\lambda_1^{(1)} n_1(x,\eta^{(1)}) + \lambda_2^{(1)} n_3(x,\eta^{(1)})$
	$(2,1)$	$r^{(1)} - r^{(2)}$
	$(0,0)$	1

TABLE 2.1

transition		rate
$(0,3) \longrightarrow$	$(1,1)$	$\lambda_1^{(1)} n_1(x,\eta^{(1)}) + \lambda_2^{(1)} n_3(x,\eta^{(1)})$
	$(0,1)$	$(1-\lambda_1^{(1)}) n_1(x,\eta^{(1)}) + (1-\lambda_2^{(1)}) n_3(x,\eta^{(1)})$
	$(2,2)$	1
	$(2,3)$	$r^{(1)} - 1$

TABLE 2.2

transition		rate
$(1,0) \longrightarrow$	$(1,1)$	$\lambda_1^{(2)} n_1(x,\eta^{(2)}) + \lambda_2^{(2)} n_3(x,\eta^{(2)})$
	$(3,2)$	$r^{(2)}$
	$(3,0)$	$r^{(1)} - r^{(2)}$
	$(0,0)$	1
$(0,2) \longrightarrow$	$(1,3)$	$\lambda_1^{(1)} n_1(x,\eta^{(1)}) + \lambda_2^{(1)} n_3(x,\eta^{(1)})$
	$(0,3)$	$\lambda_1^{(2)} n_1(x,\eta^{(2)}) - \lambda_1^{(1)} n_1(x,\eta^{(1)}) + \lambda_2^{(2)} n_3(x,\eta^{(2)}) - \lambda_2^{(1)} n_3(x,\eta^{(1)})$
	$(0,0)$	1
	$(2,2)$	$r^{(1)}$
$(1,2) \longrightarrow$	$(1,3)$	$\lambda_1^{(2)} n_1(x,\eta^{(2)}) + \lambda_2^{(2)} n_3(x,\eta^{(2)})$
	$(3,2)$	$r^{(1)}$
	$(1,0)$	1
	$(0,2)$	1
$(1,3) \longrightarrow$	$(0,2)$	1
	$(3,3)$	$r^{(1)}$
	$(1,1)$	1
$(3,0) \longrightarrow$	$(1,0)$	1
	$(2,0)$	1
	$(3,1)$	$\lambda_1^{(2)} n_1(x,\eta^{(2)}) + \lambda_2^{(2)} n_3(x,\eta^{(2)})$
	$(3,2)$	$r^{(2)}$
$(3,2) \longrightarrow$	$(3,3)$	$\lambda_1^{(2)} n_1(x,\eta^{(2)}) + \lambda_2^{(2)} n_3(x,\eta^{(2)})$
	$(1,0)$	1
	$(2,2)$	1

TABLE 2.3

To verify all the rates above are well defined, one decomposes $n_1(x,\eta^{(i)})$ and $n_3(x,\eta^{(i)})$, $(i=1,2)$, as follows

$$\begin{aligned} n_1(x,\eta^{(2)}) &= |\{y \sim x : \eta^{(2)}(y) = \eta^{(1)}(y) = 1\}| \\ &+ |\{y \sim x : \eta^{(2)}(y) = 1, \eta^{(1)}(y) = 3\}| + |\{y \sim x : \eta^{(2)}(y) = 1, \eta^{(1)}(y) \in \{0,2\}\}|, \end{aligned}$$

$$\begin{aligned} n_3(x,\eta^{(2)}) = |\{y \sim x : \eta^{(2)}(y) = \eta^{(1)}(y) = 3\}| \\ + |\{y \sim x : \eta^{(2)}(y) = 3, \eta^{(1)}(y) \in \{0,2\}\}|, \end{aligned}$$

$$n_1(x,\eta^{(1)}) = |\{y \sim x : \eta^{(2)}(y) = \eta^{(1)}(y) = 1\}|$$

$$\begin{aligned} n_3(x,\eta^{(1)}) = |\{y \sim x : \eta^{(2)}(y) = \eta^{(1)}(y) = 3\}| \\ + |\{y \sim x : \eta^{(2)}(y) = 1, \eta^{(1)}(y) = 3\}|, \end{aligned}$$

in which case, we decompose the rate

$$\lambda_1^{(2)} n_1(x,\eta^{(2)}) - \lambda_1^{(1)} n_1(x,\eta^{(1)}) + \lambda_2^{(2)} n_3(x,\eta^{(2)}) - \lambda_2^{(1)} n_3(x,\eta^{(1)})$$

$$
\begin{aligned}
&= (\lambda_1^{(2)} - \lambda_1^{(1)})|\{y \sim x : \eta^{(2)}(y) = \eta^{(1)}(y) = 1\}| \\
&+ (\lambda_1^{(2)} - \lambda_2^{(1)})|\{y \sim x : \eta^{(2)}(y) = 1, \eta^{(1)}(y) = 3\}| \\
&+ (\lambda_2^{(2)} - \lambda_2^{(1)})|\{y \sim x : \eta^{(2)}(y) = \eta^{(1)}(y) = 3\}| \\
&+ \lambda_1^{(2)}|\{y \sim x : \eta^{(2)}(y) = 1, \eta^{(1)}(y) \in \{0,2\}\}| \\
&+ \lambda_2^{(2)}|\{y \sim x : \eta^{(2)}(y) = 3, \eta^{(1)}(y) \in \{0,2\}\}|
\end{aligned} \tag{2.4.6}
$$

which is non-negative under conditions 1. to 4. coming from (I) and (III) in inequalities (2.4.4)-(2.4.5).

Rates of Table 2.2 are non-negative thanks to conditions 6. to 8., given by inequalities (II)-(i)(ii) with $\beta = B, \delta = C$. Condition 5. is used by Tables 2.1 and 2.3 that correspond to a basic coupling while Table 2.2 uses a different coupling. Table 2.3 is listing transitions of the coupled process starting from configurations that do not preserve the defined partial order, nevertheless, starting from an initial configuration where it does, dynamics of the coupling given by Tables 2.1 and 2.2 do not reach states of Table 2.3.

For a coupled process $(\eta_t^{(1)}, \eta_t^{(2)})_{t\geqslant 0}$ starting from an initial configuration such that $\eta_0^{(1)} \leqslant \eta_0^{(2)}$, since transitions of the two first Tables preserve the order on F, the markovian coupling we just constructed is increasing :

$$
\widetilde{\mathbb{P}}^{(\eta_0^{(1)},\eta_0^{(2)})}(\eta_t^{(1)} \leqslant \eta_t^{(2)}) = 1 \text{ for all } t > 0 \tag{2.4.7}
$$

where $\widetilde{\mathbb{P}}^{(\eta_0^{(1)},\eta_0^{(2)})}$ stands for the distribution of $(\eta_t^{(1)}, \eta_t^{(2)})_{t\geqslant 0}$ starting from the initial configuration $(\eta_0^{(1)}, \eta_0^{(2)})$. $\square$

We can wonder if there exists an other order than (2.2.13) for which this statement (and the following ones as well) holds. By trying out other orders in inequalities (2.2.15) of Theorem 2.2.4 , we deduce that the one defined by (2.2.13) is the only order possible here to preserve the stochastic order.

After having obtained necessary and sufficient conditions, we investigate sufficient conditions only, with which we shall work subsequently.

Proposition 2.4.2. *The symmetric process $(\eta_t)_{t\geqslant 0}$ is monotone, in the sense that, one can construct on a same probability space two symmetric processes $(\eta_t^{(1)})_{t\geqslant 0}$ and $(\eta_t^{(2)})_{t\geqslant 0}$ with respective parameters $(\lambda_1^{(1)}, \lambda_2^{(1)}, r^{(1)})$ and$(\lambda_1^{(2)}, \lambda_2^{(2)}, r^{(2)})$ satisfying $\eta_0^{(1)}, \eta_0^{(2)} \in \{0,1\}^{\mathbb{Z}^d}$, such that*

$$
\eta_0^{(1)} \leqslant \eta_0^{(2)} \Longrightarrow \eta_t^{(1)} \leqslant \eta_t^{(2)} \textit{ for all } t \geqslant 0 \textit{ a.s.} \tag{2.4.8}
$$

if all parameters satisfy

1. $\lambda_2^{(1)} \leqslant \lambda_1^{(1)}$, *3.* $\lambda_1^{(1)} \leqslant \lambda_1^{(2)}$, *5.* $r^{(1)} \geqslant r^{(2)}$
2. $\lambda_2^{(2)} \leqslant \lambda_1^{(2)}$, *4.* $\lambda_2^{(1)} \leqslant \lambda_2^{(2)}$,

Proof. Given our initial conditions, possible states for the coupled process keep laying in Table 2.1 of Proposition 2.4.1 and the coupled process does not reach any state of Tables 2.2 and 2.3. One can therefore omit conditions 4. to 6. of the previous Proposition 2.4.1 and transition rates from the couple $(0,3)$ can be defined through a basic coupling even if it does not preserve the order :

transition	rate
$(0,3) \longrightarrow (1,3)$	$\lambda_1^{(1)} n_1(x,\eta^{(1)}) + \lambda_2^{(1)} n_3(x,\eta^{(1)})$
$(0,3) \longrightarrow (0,1)$	1
$(0,3) \longrightarrow (0,2)$	1
$(0,3) \longrightarrow (2,3)$	$r^{(1)}$

Table 2.4

in which case, Table 2.4 substitutes Table 2.2. □

Remark 2.4.1. *In view of the proof of Proposition 2.4.2, one can actually relax the admissible initial conditions : it is enough to assume* $\eta_0^{(1)}$ *and* $\eta_0^{(2)}$ *satisfying* $\eta_0^{(1)} \leqslant \eta_0^{(2)}$ *and for all* $x \in \mathbb{Z}^d$, $(\eta_0^{(1)}(x), \eta_0^{(2)}(x)) \neq (0,3)$. *In particular one could start from* $\eta_0^{(1)} = \eta_0^{(2)}$.

Since Tables 2.1, 2.4 and 2.3 correspond to a basic coupling, to construct such processes on a same probability space via the graphical representation, one define from Section 1.2 mutually independent Poisson processes : $\{T_n^{x,y}, n \geqslant 1\}$ with rate $\lambda_1^{(2)}$, $\{D_n^{1,x}, n \geqslant 1\}$ with rate 1, $\{D_n^{2,x}, n \geqslant 1\}$ with rate 1 and independent uniform random variables $\{U_n^{x,y}, n \geqslant 1\}$ on $(0,1)$, independent of the Poisson processes. Indeed, after conditions 1 to 5 of Proposition 2.4.2, the growth rate $\lambda_1^{(2)}$ is the largest one. At each time $t = T_n^{x,y}$ a birth might occur and the uniform random variables determine if it occurs or not. For instance, if $\eta_t^{(1)}(x)$ is in state 3 and $\eta_t^{(1)}(y)$ is in state 0, then a birth in y for $\eta_t^{(1)}$ occurs if $U_n^{x,y} < \lambda_2^{(1)}/\lambda_1^{(2)}$; if $\eta_t^{(2)}(x)$ is in state 3 and $\eta_t^{(2)}(y)$ is in state 0, then a birth in y for the process $\eta_t^{(2)}$ occurs if $U_n^{x,y} < \lambda_2^{(2)}/\lambda_1^{(2)}$. Since $\lambda_2^{(1)} \leqslant \lambda_2^{(2)}$, as soon as $U_n^{x,y} \in (\lambda_2^{(1)}/\lambda_1^{(2)}, \lambda_2^{(2)}/\lambda_1^{(2)}]$ an arrow used by the process $\eta_t^{(2)}$ is not used by the process $\eta_t^{(1)}$.

If $(\eta_t^{(1)})_{t\geqslant 0}$ and $(\eta_t^{(2)})_{t\geqslant 0}$ differ by at most one parameter, one deduces from Proposition 2.4.2 several monotonicity properties :

Corollary 2.4.1. *Suppose* $\eta_0^{(1)}, \eta_0^{(2)} \in \{0,1\}^{\mathbb{Z}^d}$. *Then for the processes* $(\eta_t^{(1)})_{t\geqslant 0}$ *and* $(\eta_t^{(2)})_{t\geqslant 0}$ *with parameters* $(\lambda_1^{(1)}, \lambda_2^{(1)}, r^{(1)})$ *and* $(\lambda_1^{(2)}, \lambda_2^{(2)}, r^{(2)})$ *respectively, one has*

(i) Attractiveness : if $(\lambda_1^{(1)}, \lambda_2^{(1)}, r^{(1)}) = (\lambda_1^{(2)}, \lambda_2^{(2)}, r^{(2)})$ *, then* $\eta_0^{(1)} \leqslant \eta_0^{(2)} \Longrightarrow \eta_t^{(1)} \leqslant \eta_t^{(2)}$ *a.s., for all* $t \geqslant 0$.

(ii) *Increase w.r.t.* λ_1 *: if* $(\eta_t^{(1)})_{t\geqslant 0}$ *and* $(\eta_t^{(2)})_{t\geqslant 0}$ *have respective parameters* $(\lambda_1^{(1)}, \lambda_2, r)$ *and* $(\lambda_1^{(2)}, \lambda_2, r)$ *such that* $\lambda_2 \leqslant \lambda_1^{(1)} \leqslant \lambda_1^{(2)}$*, then* $\eta_0^{(1)} \leqslant \eta_0^{(2)} \Longrightarrow \eta_t^{(1)} \leqslant \eta_t^{(2)}$ *a.s., for all* $t \geqslant 0$.

(iii) *Increase w.r.t.* λ_2 *: if* $(\eta_t^{(1)})_{t\geqslant 0}$ *and* $(\eta_t^{(2)})_{t\geqslant 0}$ *have respective parameters* $(\lambda_1, \lambda_2^{(1)}, r)$ *and* $(\lambda_1, \lambda_2^{(2)}, r)$ *such that* $\lambda_2^{(1)} \leqslant \lambda_2^{(2)} \leqslant \lambda_1$*, then* $\eta_0^{(1)} \leqslant \eta_0^{(2)} \Longrightarrow \eta_t^{(1)} \leqslant \eta_t^{(2)}$ *a.s., for all* $t \geqslant 0$.

(iv) *Decrease w.r.t.* r *: if* $(\eta_t^{(1)})_{t\geqslant 0}$ *and* $(\eta_t^{(2)})_{t\geqslant 0}$ *have respective parameters* $(\lambda_1, \lambda_2, r^{(1)})$ *and* $(\lambda_1, \lambda_2, r^{(2)})$ *such that* $r^{(1)} \geqslant r^{(2)}$ *with* $\lambda_2 < \lambda_1$*, then* $\eta_0^{(1)} \leqslant \eta_0^{(2)} \Longrightarrow \eta_t^{(1)} \leqslant \eta_t^{(2)}$ *a.s., for all* $t \geqslant 0$.

A consequence related to Corollary (2.4.1)-(iv) is the non-increase of the survival probability with respect to the release rate r for fixed λ_1, λ_2 :

Corollary 2.4.2. *Suppose* λ_2 *and* λ_1 *fixed. If* $(\eta_t)_{t\geqslant 0}$ *has initial configuration* $\eta_0 = \mathbf{1}_{\{0\}}$*, the mapping*

$$r \longmapsto \mathbb{P}_r(\forall t \geqslant 0,\ H_t \neq \varnothing)$$

is a non-increasing function.

Proof. Indeed if $(\eta_t^{(1)})_{t\geqslant 0}$ and $(\eta_t^{(2)})_{t\geqslant 0}$ are two processes such that $\eta_0^{(1)}, \eta_0^{(2)} \in \{0,1\}^{\mathbb{Z}^d}$ and with respective parameters $(\lambda_1, \lambda_2, r^{(1)})$ and $(\lambda_1, \lambda_2, r^{(2)})$ such that $r^{(1)} \leqslant r^{(2)}$, then according to Corollary 2.4.1,

$$H_0^{(2)} \subset H_0^{(1)} \Longrightarrow H_t^{(2)} \subset H_t^{(1)},$$

for all $t \geqslant 0$. $\square$

One defined the asymmetric process as a particular case of the symmetric process where the transition from state 2 to state 3 does not occur. One can thus, in a similar way to Propositions 2.4.1 and 2.4.2, obtain necessary and sufficient conditions, then, only sufficient conditions, for the monotonicity of the asymmetric process.

Proposition 2.4.3. *The asymmetric process* $(\eta_t)_{t\geqslant 0}$ *is monotone in the sense that, conditions*

1. $\lambda_2^{(1)} \leqslant \lambda_1^{(1)}$,
2. $\lambda_2^{(2)} \leqslant \lambda_1^{(2)}$,
3. $\lambda_1^{(1)} \leqslant \lambda_1^{(2)}$,
4. $\lambda_2^{(1)} \leqslant \lambda_2^{(2)}$,
5. $r^{(1)} \geqslant r^{(2)}$,

are sufficient to construct on a same probability space two asymmetric process $(\eta_t^{(1)})_{t\geqslant 0}$ *and* $(\eta_t^{(2)})_{t\geqslant 0}$ *with respective parameters* $(\lambda_1^{(1)}, \lambda_2^{(1)}, r^{(1)})$ *and* $(\lambda_1^{(2)}, \lambda_2^{(2)}, r^{(2)})$ *and with initial condition* $\eta_0^{(1)}, \eta_0^{(2)} \in \{0,1\}^{\mathbb{Z}^d}$*, such that*

$$\eta_0^{(1)} \leqslant \eta_0^{(2)} \Longrightarrow \eta_t^{(1)} \leqslant \eta_t^{(2)} \text{ a.s., for all } t \geqslant 0. \tag{2.4.9}$$

Proof. As in the proof of Proposition 2.4.2, one applies Theorem 2.2.4 with $j_1, h_1 \in \{0,1\}$ to two asymmetric processes $(\eta_t^{(1)})_{t\geqslant 0}$ and $(\eta_t^{(2)})_{t\geqslant 0}$ with respective parameters $(\lambda_1^{(1)}, \lambda_2^{(1)}, r^{(1)})$ and $(\lambda_1^{(2)}, \lambda_2^{(2)}, r^{(2)})$. Using relations (2.4.2)-(2.4.3) with the corresponding rates of both processes given by (2.2.16), with $(\alpha,\beta) \leqslant (\gamma,\delta)$, one has

$$\begin{aligned}
&\mathbf{1}\{j_1=0\}\Big(\mathbf{1}\{k=2\}\Big(\mathbf{1}\{\beta=\delta=B\}\big(\lambda_1^{(1)}\mathbf{1}\{\alpha=D\}+\lambda_2^{(1)}\mathbf{1}\{\alpha=C\}\big)\Big)\\
&\qquad+\mathbf{1}\{k=1\}\Big(\mathbf{1}\{\beta=\delta=A\}+\mathbf{1}\{\beta=\delta=C\}\Big)\Big)\\
&+\mathbf{1}\{j_1=0\}\mathbf{1}\{k=2\}\Big(\mathbf{1}\{\beta=B,\delta=C\}\big(\lambda_1^{(1)}\mathbf{1}\{\alpha=D\}+\lambda_2^{(1)}\mathbf{1}\{\alpha=C\}\big)\Big)\\
&+\mathbf{1}\{j_1=1\}\mathbf{1}\{k=2\}\Big(\mathbf{1}\{\beta=\delta=B\}\big(\lambda_1^{(1)}\mathbf{1}\{\alpha=D\}+\lambda_2^{(1)}\mathbf{1}\{\alpha=C\}\big)\Big)\\
&\leqslant \mathbf{1}\{j_1=0\}\Big(\mathbf{1}\{l=2\}\mathbf{1}\{\delta=B\}\big(\lambda_1^{(2)}\mathbf{1}\{\gamma=D\}+\lambda_2^{(2)}\mathbf{1}\{\gamma=C\}\big)\\
&\qquad+\mathbf{1}\{l=1\}\big(\mathbf{1}\{\delta=A\}+\mathbf{1}\{\delta=C\}\big)\Big)\\
&+\mathbf{1}\{j_1=1\}\Big(\mathbf{1}\{l=2\}\mathbf{1}\{\delta=B\}\big(\lambda_1^{(2)}\mathbf{1}\{\gamma=D\}+\lambda_2^{(2)}\mathbf{1}\{\gamma=C\}\big)
\end{aligned}\tag{2.4.10}$$

while the second relation leaves (2.4.5) unchanged. One deduces the following necessary conditions :

(I) $j_1 \in \{0,1\}$, $\delta=\beta=B$ in (2.4.10) give
  (i) $\alpha=\gamma=C$, $\beta=B$, $\delta=C$: $\lambda_2^{(1)} \leqslant \lambda_2^{(2)}$ stated by condition 4.
  (ii) $\alpha=C, \gamma=D$: $\lambda_2^{(1)} \leqslant \lambda_1^{(2)}$ stated by conditions 1. and 3.
  (iii) $\alpha=\gamma=D$: $\lambda_1^{(1)} \leqslant \lambda_1^{(2)}$ stated by condition 3.

(II) $j_1=0$, $\beta=B$, $\delta=1+\beta=C$ in (2.4.10) give
  (i) $\alpha=D$: $\lambda_1^{(1)} \leqslant 1$.
  (ii) $\alpha=C$: $\lambda_2^{(1)} \leqslant 1$.

The relation (2.4.5) staying unchanged, one has
(III) $h_1=0$, $\gamma=\alpha\in\{B,D\}$ in (2.4.5) give $r^{(1)} \geqslant r^{(2)}$ stated by condition 5.
(IV) $h_1=0$,$\alpha=B, \gamma=1+\alpha=C$ in (2.4.5) give $r^{(1)} \geqslant 1$.
The obtained necessary conditions are

1. $\lambda_2^{(1)} \leqslant \lambda_1^{(1)}$,
2. $\lambda_2^{(2)} \leqslant \lambda_1^{(2)}$,
3. $\lambda_1^{(1)} \leqslant \lambda_1^{(2)}$,
4. $\lambda_2^{(1)} \leqslant \lambda_2^{(2)}$,
5. $r^{(1)} \geqslant r^{(2)}$
6. $\lambda_1^{(1)} \leqslant 1$,
7. $\lambda_2^{(1)} \leqslant 1$,
8. $r^{(1)} \geqslant 1$.

As for Proposition 2.4.1, these conditions allow us to construct an increasing markovian coupling. As in Proposition 2.4.2, given our initial configurations, state $(0,3)$ is not possible for the coupled process. One can thus dispense conditions 6 to 8. and sufficient conditions to settle an increasing markovian coupling as in Proposition 2.4.2 are given by

1. $\lambda_2^{(1)} \leqslant \lambda_1^{(1)}$,
2. $\lambda_2^{(2)} \leqslant \lambda_1^{(2)}$,
3. $\lambda_1^{(1)} \leqslant \lambda_1^{(2)}$,
4. $\lambda_2^{(1)} \leqslant \lambda_2^{(2)}$,
5. $r^{(1)} \geqslant r^{(2)}$

Details of the dynamics of the coupled process $(\eta_t^{(1)}, \eta_t^{(2)})_{t\geqslant 0}$ are summarized in the following tables.

transition		rate
$(0,0) \longrightarrow$	$(1,1)$	$\lambda_1^{(1)} n_1(x,\eta^{(1)}) + \lambda_2^{(1)} n_3(x,\eta^{(1)})$
	$(0,1)$	$\lambda_1^{(2)} n_1(x,\eta^{(2)}) - \lambda_1^{(1)} n_1(x,\eta^{(1)}) + \lambda_2^{(2)} n_3(x,\eta^{(2)}) - \lambda_2^{(1)} n_3(x,\eta^{(1)})$
	$(2,2)$	$r^{(2)}$
	$(2,0)$	$r^{(1)} - r^{(2)}$
$(1,1) \longrightarrow$	$(0,0)$	1
	$(3,3)$	$r^{(2)}$
	$(3,1)$	$r^{(1)} - r^{(2)}$
$(2,2) \longrightarrow$	$(0,0)$	1
$(3,3) \longrightarrow$	$(1,1)$	1
	$(2,2)$	1
$(2,0) \longrightarrow$	$(2,1)$	$\lambda_1^{(2)} n_1(x,\eta^{(2)}) + \lambda_2^{(2)} n_3(x,\eta^{(2)})$
	$(0,0)$	1
	$(2,2)$	$r^{(2)}$
$(2,3) \longrightarrow$	$(0,1)$	1
	$(2,2)$	1
$(2,1) \longrightarrow$	$(2,0)$	1
	$(0,1)$	1
	$(2,3)$	$r^{(2)}$
$(3,1) \longrightarrow$	$(2,0)$	1
	$(1,1)$	1
	$(3,3)$	$r^{(2)}$
$(0,1) \longrightarrow$	$(2,3)$	$r^{(2)}$
	$(1,1)$	$\lambda_1^{(1)} n_1(x,\eta^{(1)}) + \lambda_2^{(1)} n_3(x,\eta^{(1)})$
	$(2,1)$	$r^{(1)} - r^{(2)}$
	$(0,0)$	1

TABLE 2.5

transition		rate
$(0,3) \longrightarrow$	$(1,3)$	$\lambda_1^{(1)} n_1(x,\eta^{(1)}) + \lambda_2^{(1)} n_3(x,\eta^{(1)})$
	$(0,1)$	1
	$(0,2)$	1
	$(2,3)$	$r^{(1)}$

TABLE 2.6

transition	rate
$(1,0) \longrightarrow (1,1)$	$\lambda_1^{(2)} n_1(x,\eta^{(2)}) + \lambda_2^{(2)} n_3(x,\eta^{(2)})$
$(1,0) \longrightarrow (3,2)$	$r^{(2)}$
$(1,0) \longrightarrow (3,0)$	$r^{(1)} - r^{(2)}$
$(1,0) \longrightarrow (0,0)$	1
$(0,2) \longrightarrow (1,2)$	$\lambda_1^{(1)} n_1(x,\eta^{(1)}) + \lambda_2^{(1)} n_3(x,\eta^{(1)})$
$(0,2) \longrightarrow (0,0)$	1
$(0,2) \longrightarrow (2,2)$	$r^{(1)}$
$(1,2) \longrightarrow (3,2)$	$r^{(1)}$
$(1,2) \longrightarrow (1,0)$	1
$(1,2) \longrightarrow (0,2)$	1
$(1,3) \longrightarrow (0,2)$	1
$(1,3) \longrightarrow (3,3)$	$r^{(1)}$
$(1,3) \longrightarrow (1,1)$	1
$(3,0) \longrightarrow (1,0)$	1
$(3,0) \longrightarrow (2,0)$	1
$(3,0) \longrightarrow (3,1)$	$\lambda_1^{(2)} n_1(x,\eta^{(2)}) + \lambda_2^{(2)} n_3(x,\eta^{(2)})$
$(3,0) \longrightarrow (3,2)$	$r^{(2)}$
$(3,2) \longrightarrow (1,0)$	1
$(3,2) \longrightarrow (2,2)$	1

TABLE 2.7

Remark now that Tables 2.5 and 2.7 differ from Tables 2.1 and 2.3 but Table 2.6 stays identical to Table 2.4. As for Proposition 2.4.2, under conditions 1. to 5., if the initial conditions given by $\eta_0^{(1)}$ and $\eta_0^{(2)}$ satisfy $\eta_0^{(1)} \leqslant \eta_0^{(2)}$ and $\eta_0^{(1)}, \eta_0^{(2)} \in \{0,1\}^{\mathbb{Z}^d}$, this markovian coupling is increasing since the transitions of the coupled process belong to Table 2.5 and

$$\widetilde{\mathbb{P}}^{(\eta_0^{(1)},\eta_0^{(2)})}(\eta_t^{(1)} \leqslant \eta_t^{(2)}) = 1 \text{ for all } t > 0 \tag{2.4.11}$$

where $\widetilde{\mathbb{P}}^{(\eta_0^{(1)},\eta_0^{(2)})}$ denotes the distribution of $(\eta_t^{(1)}, \eta_t^{(2)})_{t\geqslant 0}$ starting from the initial configuration $(\eta_0^{(1)}, \eta_0^{(2)})$. □

One can compare the symmetric with the asymmetric process as well.

Proposition 2.4.4. *Let $(\eta_t)_{t\geqslant 0}$ be an asymmetric process and $(\chi_t)_{t\geqslant 0}$ be a symmetric process, both with parameters $(\lambda_1, \lambda_2, r)$ and $\eta_0, \chi_0 \in \{0,1\}^{\mathbb{Z}^d}$ such that $\lambda_2 < \lambda_1$, then for all $t \geqslant 0$,*

$$\eta_0 \leqslant \chi_0 \Rightarrow \eta_t \leqslant \chi_t \text{ a.s. for all } t \geqslant 0$$

Proof. Apply Theorem 2.2.4 with an asymmetric process $(\eta_t)_{t\geqslant 0}$ and a symmetric process $(\chi_t)_{t\geqslant 0}$ with parameters $(\lambda_1, \lambda_2, r)$. Necessary and sufficient conditions on the parameters to obtain a stochastic order are given by (2.4.2)-(2.4.3) that become

$$\begin{aligned}
&\mathbf{1}\{j_1 = 0\}\mathbf{1}\{k = 2\}\mathbf{1}\{\beta = \delta = B\}\big(\lambda_1\mathbf{1}\{\alpha = D\} + \lambda_2\mathbf{1}\{\alpha = C\}\big)\\
&+ \mathbf{1}\{j_1 = 0\}\mathbf{1}\{k = 2\}\mathbf{1}\{\beta = B, \delta = C\}\big(\lambda_1\mathbf{1}\{\alpha = D\} + \lambda_2\mathbf{1}\{\alpha = C\}\big)\\
&+ \mathbf{1}\{j_1 = 0\}\mathbf{1}\{k = 1\}\Big(\mathbf{1}\{\beta = \delta = A\} + \mathbf{1}\{\beta = \delta = C\}\Big)\\
&+ \mathbf{1}\{j_1 = 1\}\mathbf{1}\{k = 2\}\mathbf{1}\{\beta = \delta = B\}\big(\lambda_1\mathbf{1}\{\alpha = D\} + \lambda_2\mathbf{1}\{\alpha = C\}\big)
\end{aligned}$$

$$
\begin{aligned}
&\leqslant \mathbf{1}\{j_1=0\}\Big(\mathbf{1}\{l=2\}\big(\mathbf{1}\{\delta=B\}+\mathbf{1}\{\delta=A\}\big)\big(\lambda_1\mathbf{1}\{\gamma=D\}+\lambda_2\mathbf{1}\{\gamma=C\}\big)\\
&\qquad +\mathbf{1}\{l=1\}\big(\mathbf{1}\{\delta=A\}+\mathbf{1}\{\delta=C\}\big)\Big)\\
&\quad +\mathbf{1}\{j_1=1\}\mathbf{1}\{l=2\}\big(\mathbf{1}\{\delta=B\}+\mathbf{1}\{\delta=A\}\big)\big(\lambda_1\mathbf{1}\{\gamma=D\}+\lambda_2\mathbf{1}\{\gamma=C\}\big)
\end{aligned}
\tag{2.4.12}
$$

and

$$
\begin{aligned}
&\mathbf{1}\{h_1=0\}\Big(\mathbf{1}\{k=1\}r(\mathbf{1}\{\alpha=B\}+\mathbf{1}\{\alpha=D\})+\mathbf{1}\{k=2\}(\mathbf{1}\{\alpha=C\}\\
&\qquad +\mathbf{1}\{\alpha=D\})\Big)+\mathbf{1}\{h_1=1\}\Big(\mathbf{1}\{k=2\}(\mathbf{1}\{\alpha=C\}+\mathbf{1}\{\alpha=D\})\Big)\\
&\geqslant \mathbf{1}\{h_1=0\}\mathbf{1}\{\gamma=\alpha\}\Big(\mathbf{1}\{l=1\}r(\mathbf{1}\{\gamma=B\}+\mathbf{1}\{\gamma=D\})\\
&\qquad +\mathbf{1}\{l=2\}(\mathbf{1}\{\gamma=C\}+\mathbf{1}\{\gamma=D\})\Big)\\
&\quad +\mathbf{1}\{h_1=0\}\mathbf{1}\{\gamma=1+\alpha\}\Big(\mathbf{1}\{l=2\}(\mathbf{1}\{\gamma=C\}+\mathbf{1}\{\gamma=D\})\Big)\\
&\quad +\mathbf{1}\{h_1=1\}\mathbf{1}\{\gamma=\alpha\}\Big(\mathbf{1}\{l=2\}(\mathbf{1}\{\gamma=C\}+\mathbf{1}\{\gamma=D\})\Big)
\end{aligned}
\tag{2.4.13}
$$

These equations exhibit the following necessary conditions :

(I) $j_1 \in \{0,1\}$, $\delta=\beta=B$,$\alpha=C,\gamma=D$ in (2.4.12) give : $\lambda_2 \leqslant \lambda_1$

(II) $h_1=0$, $\alpha=B,\gamma=1+\alpha=C$ in (2.4.13) give $r \geqslant 1$

As previously, condition $r \geqslant 1$ is necessary to construct an increasing markovian coupled process in a general framework, but if one restricts the initial conditions to satisfy $\eta_0 \leqslant \chi_0$ and $\eta_0,\chi_0 \in \{0,1\}^{\mathbb{Z}^d}$, condition $\lambda_2 \leqslant \lambda_1$ is sufficient and the coupled process can be constructed through the following transitions :

transition	rate
$(0,0) \longrightarrow (1,1)$	$\lambda_1 n_1(x,\eta) + \lambda_2 n_3(x,\eta)$
$(0,0) \longrightarrow (0,1)$	$\lambda_1(n_1(x,\chi) - n_1(x,\eta)) + \lambda_2 n_3(x,\chi) - n_3(x,\eta))$
$(0,0) \longrightarrow (2,2)$	r
$(1,1) \longrightarrow (0,0)$	1
$(1,1) \longrightarrow (3,3)$	r
$(2,2) \longrightarrow (0,0)$	1
$(2,2) \longrightarrow (2,3)$	$\lambda_1 n_1(x,\chi) + \lambda_2 n_3(x,\chi)$
$(3,3) \longrightarrow (1,1)$	1
$(3,3) \longrightarrow (2,2)$	1
$(2,0) \longrightarrow (2,1)$	$\lambda_1 n_1(x,\chi) + \lambda_2 n_3(x,\chi)$
$(2,0) \longrightarrow (0,0)$	1
$(2,0) \longrightarrow (2,2)$	r
$(2,3) \longrightarrow (0,1)$	1
$(2,3) \longrightarrow (2,2)$	1
$(2,1) \longrightarrow (2,0)$	1
$(2,1) \longrightarrow (0,1)$	1
$(2,1) \longrightarrow (2,3)$	r
$(3,1) \longrightarrow (2,0)$	1
$(3,1) \longrightarrow (1,1)$	1
$(3,1) \longrightarrow (3,3)$	r
$(0,1) \longrightarrow (2,3)$	r
$(0,1) \longrightarrow (1,1)$	$\lambda_1 n_1(x,\eta) + \lambda_2 n_3(x,\eta)$
$(0,1) \longrightarrow (0,0)$	1

TABLE 2.8

transition	rate
$(0,3) \longrightarrow (2,3)$	r
$(0,3) \longrightarrow (1,3)$	$\lambda_1 n_1(x,\eta) + \lambda_2 n_3(x,\eta)$
$(0,3) \longrightarrow (0,1)$	1
$(0,3) \longrightarrow (0,2)$	1

TABLE 2.9

transition		rate
$(1,0) \longrightarrow$	$(1,1)$	$\lambda_1 n_1(x,\chi) + \lambda_2 n_3(x,\chi)$
	$(3,2)$	r
	$(0,0)$	1
$(0,2) \longrightarrow$	$(1,3)$	$\lambda_1 n_1(x,\eta) + \lambda_2 n_3(x,\eta)$
	$(0,3)$	$\lambda_1 n_1(x,\chi) + \lambda_2 n_3(x,\chi)$
	$(0,0)$	1
	$(2,2)$	r
$(1,2) \longrightarrow$	$(3,2)$	r
	$(0,0)$	1
	$(1,3)$	$\lambda_1 n_1(x,\chi) + \lambda_2 n_3(x,\chi)$
$(1,3) \longrightarrow$	$(0,2)$	1
	$(3,3)$	r
$(3,0) \longrightarrow$	$(1,0)$	1
	$(2,0)$	1
	$(3,1)$	$\lambda_1 n_1(x,\chi) + \lambda_2 n_3(x,\chi)$
	$(3,2)$	r

TABLE 2.10

As in (2.4.6), the second rate in Table 2.8 is positive. Starting from an initial configuration such that $\eta_0 \leqslant \chi_0$ and $\eta_0, \chi_0 \in \{0,1\}^{\mathbb{Z}^d}$, the coupled process does not reach any configuration of Tables 2.9 and 2.10. Condition $\lambda_2 \leqslant \lambda_1$ is sufficient to obtain

$$\eta_t \leqslant \chi_t \text{ a.s., for all } t \geqslant 0.$$

□

Finally, one settles two comparisons between a basic contact process and a multitype process.

Proposition 2.4.5. *Let $(\xi_t)_{t\geqslant 0}$ be a basic contact process on $\{0,1\}^{\mathbb{Z}^d}$ with growth rate λ_1 and let $(\chi_t)_{t\geqslant 0}$ be a symmetric multitype process with parameters $(\lambda_1, \lambda_2, r)$ such that $\lambda_2 < \lambda_1$. Then,*

$$\chi_0 \leqslant \xi_0 \Rightarrow \chi_t \leqslant \xi_t \textit{ a.s. for all } t \geqslant 0$$

Proof. Consider the basic contact process $(\xi_t)_{t\geqslant 0}$ viewed as a symmetric multitype process with parameters $(\lambda_1^{(2)}, \lambda_2^{(2)}, r^{(2)})$ with $\lambda_1^{(2)} = \lambda_1$, $\lambda_2^{(2)} = 0$, $r^{(2)} = 0$. Values A and C do not exist for the process ξ_t, retrieving the proof of Proposition 2.4.1, relations (2.4.2)-(2.4.3) become

$$\begin{aligned}
&\mathbf{1}\{j_1 = 0\}\mathbf{1}\{k=2\}\mathbf{1}\{\beta=\delta=B\}\big(\lambda_1\mathbf{1}\{\alpha=D\} + \lambda_2\mathbf{1}\{\alpha=C\}\big)\Big) \\
&+ \mathbf{1}\{j_1=0\}\mathbf{1}\{k=2\}\Big(\mathbf{1}\{\beta=A,\delta=B\}\big(\lambda_1\mathbf{1}\{\alpha=D\}+\lambda_2\mathbf{1}\{\alpha=C\}\big) \\
&\quad + \mathbf{1}\{j_1=1\}\mathbf{1}\{k=2\}\mathbf{1}\{\beta=\delta=B\}\big(\lambda_1\mathbf{1}\{\alpha=D\}+\lambda_2\mathbf{1}\{\alpha=C\}\big) \\
&\leqslant \big(\mathbf{1}\{j_1=0\}+\mathbf{1}\{j_1=1\}\big)\mathbf{1}\{l=2\}\mathbf{1}\{\delta=B\}\lambda_1\mathbf{1}\{\gamma=D\}
\end{aligned} \tag{2.4.14}$$

and

$$\begin{aligned}
&\mathbf{1}\{h_1 = 0\}\Big(\mathbf{1}\{k = 1\}r(\mathbf{1}\{\alpha = B\} + \mathbf{1}\{\alpha = D\}) + \mathbf{1}\{k = 2\}(\mathbf{1}\{\alpha = C\} \\
&\qquad + \mathbf{1}\{\alpha = D\})\Big) + \mathbf{1}\{h_1 = 1\}\mathbf{1}\{k = 2\}(\mathbf{1}\{\alpha = C\} + \mathbf{1}\{\alpha = D\}) \\
&\geqslant \mathbf{1}\{h_1 = 0\}\mathbf{1}\{\gamma = \alpha\}\mathbf{1}\{l = 2\}\mathbf{1}\{\gamma = D\} \\
&+ \mathbf{1}\{h_1 = 0\}\mathbf{1}\{\gamma = 1 + \alpha\}\mathbf{1}\{l = 2\}\mathbf{1}\{\gamma = D\} \\
&\quad + \mathbf{1}\{h_1 = 1\}\mathbf{1}\{\gamma = \alpha\}\mathbf{1}\{l = 2\}\mathbf{1}\{\gamma = D\}
\end{aligned} \tag{2.4.15}$$

that exhibit the following necessary condition : $j_1 \in \{0, 1\}$, $\beta = \delta = B$, $\alpha = C$, $\gamma = D$ in (2.4.14) give $\lambda_2 \leqslant \lambda_1$. While relation (2.4.15) does not give further condition. Condition $\lambda_2 \leqslant \lambda_1$ is sufficient and allows us to construct the following coupling.

transition		rate
$(0,0) \longrightarrow$	$(2,0)$	r
	$(1,1)$	$\lambda_1 n_1(x,\chi) + \lambda_2 n_3(x,\chi)$
	$(0,1)$	$\lambda_1(n_1(x,\xi) - n_1(x,\chi)) - \lambda_2 n_3(x,\chi)$
$(1,1) \longrightarrow$	$(0,0)$	1
	$(3,1)$	r
$(2,0) \longrightarrow$	$(3,1)$	$\lambda_1 n_1(x,\chi) + \lambda_2 n_3(x,\chi)$
	$(2,1)$	$\lambda_1(n_1(x,\xi) - n_1(x,\chi)) - \lambda_2 n_3(x,\chi)$
	$(0,0)$	1
$(2,1) \longrightarrow$	$(2,0)$	1
	$(0,1)$	1
	$(3,1)$	$\lambda_1 n_1(x,\chi) + \lambda_2 n_3(x,\chi)$
$(3,1) \longrightarrow$	$(2,0)$	1
	$(1,1)$	1
$(0,1) \longrightarrow$	$(1,1)$	$\lambda_1 n_1(x,\chi) + \lambda_2 n_3(x,\chi)$
	$(2,1)$	r
	$(0,0)$	1
$(1,0) \longrightarrow$	$(1,1)$	$\lambda_1 n_1(x,\chi) + \lambda_2 n_3(x,\chi)$
	$(3,0)$	r
	$(0,0)$	1
$(3,0) \longrightarrow$	$(2,0)$	1
	$(1,0)$	1
	$(3,1)$	$\lambda_1 n_1(x,\xi)$

TABLE 2.11

For all $x \in \mathbb{Z}^d$, one has if $\chi \leqslant \xi$

$$n_1(x,\chi) = |\{y \sim x : \chi(y) = \xi(y) = 1\}| \tag{2.4.16}$$

$$n_3(x,\chi) = |\{y \sim x : \chi(y) = 3, \xi(y) = 1\}| \tag{2.4.17}$$

$$\begin{aligned} n_1(x,\xi) = |\{y \sim x : \chi(y) = \xi(y) = 1\}| + |\{y \sim x : \chi(y) = 3, \xi(y) = 1\}| \\ + |\{y \sim x : \chi(y) \in \{0,2\}, \xi(y) = 1\}| \end{aligned} \tag{2.4.18}$$

Therefore, under condition condition $\lambda_2 \leqslant \lambda_1$

$$\lambda_1(n_1(x,\xi) - n_1(x,\chi)) - \lambda_2 n_3(x,\chi) \\ = (\lambda_1 - \lambda_2)n_3(x,\chi) + \lambda_1|\{y \sim x : \chi(y) \in \{0,2\}, \xi(y) = 1\}|$$

is non-negative, and

$$\chi_0 \leqslant \xi_0 \Rightarrow \chi_t \leqslant \xi_t \ a.s.,$$

for all $t \geqslant 0$. □

For next proposition, $(\widetilde{\xi}_t)_{t\geqslant 0}$ is a basic contact process on $\{2,3\}^{\mathbb{Z}^d}$ whose dynamics is given by the following transitions in $x \in \mathbb{Z}^d$

$$2 \to 3 \text{ at rate } \lambda_2 n_3(x,\widetilde{\xi}), \quad 3 \to 2 \text{ at rate } 1 \tag{2.4.19}$$

Proposition 2.4.6. *Let $(\eta_t)_{t\geqslant 0}$ be a symmetric multitype process with parameters $(\lambda_1, \lambda_2, r)$ such that $\lambda_2 \leqslant \lambda_1$. Then*

$$\widetilde{\xi}_0 \leqslant \eta_0 \Rightarrow \widetilde{\xi}_t \leqslant \eta_t \ a.s., \text{ for all } t \geqslant 0.$$

Proof. Use once again Theorem 2.2.4 to obtain necessary and sufficient conditions for a stochastic order. For the process $(\widetilde{\xi}_t)_{t\geqslant 0}$, values B and D are not reached and rates are given by (2.2.19). Necessary and sufficient conditions on the parameters are given by relations (2.4.2)-(2.4.3) applied to rates (2.4.19) i.e. (2.2.19), (2.2.16) and (2.2.17),

$$\begin{aligned}
&\big(\mathbf{1}\{j_1 = 0\} + \mathbf{1}\{j_1 = 1\}\big)\mathbf{1}\{k = 2\}\mathbf{1}\{\beta = \delta = A\}\lambda_2\mathbf{1}\{\alpha = C\} \\
&\qquad + \mathbf{1}\{j_1 = 0\}\mathbf{1}\{k = 2\}\mathbf{1}\{\delta = B, \beta = A\}\lambda_2\mathbf{1}\{\alpha = C\} \\
&\leqslant \mathbf{1}\{j_1 = 0\}\Big(\mathbf{1}\{l = 2\}\big(\mathbf{1}\{\delta = B\} + \mathbf{1}\{\delta = A\}\big)\big(\lambda_1\mathbf{1}\{\gamma = D\} + \lambda_2\mathbf{1}\{\gamma = C\}\big) \\
&\qquad\qquad + \mathbf{1}\{l = 1\}\mathbf{1}\{\delta = A\}\Big) \\
&\quad + \mathbf{1}\{j_1 = 1\}\mathbf{1}\{l = 2\}\big(\mathbf{1}\{\delta = B\} + \mathbf{1}\{\delta = A\}\big)\big(\lambda_1\mathbf{1}\{\gamma = D\} + \lambda_2\mathbf{1}\{\gamma = C\}\big)
\end{aligned} \tag{2.4.20}$$

and

$$\begin{aligned}
&\mathbf{1}\{h_1 = 0\}\mathbf{1}\{k = 2\}\mathbf{1}\{\alpha = C\} + \mathbf{1}\{h_1 = 1\}\mathbf{1}\{k = 2\}\mathbf{1}\{\alpha = C\} \\
&\geqslant \mathbf{1}\{h_1 = 0\}\mathbf{1}\{\gamma = \alpha\}\mathbf{1}\{l = 2\}\mathbf{1}\{\gamma = C\} \\
&+ \mathbf{1}\{h_1 = 0\}\mathbf{1}\{\gamma = 1 + \alpha\}\mathbf{1}\{l = 2\}\mathbf{1}\{\gamma = D\} \\
&\quad + \mathbf{1}\{h_1 = 1\}\mathbf{1}\{\gamma = \alpha\}\mathbf{1}\{l = 2\}\mathbf{1}\{\gamma = C\}
\end{aligned} \tag{2.4.21}$$

exhibiting the following conditions : $j_1 \in \{0,1\}$, $\beta = \delta = A$, $\alpha = C$, $\gamma = D$ in (2.4.20) give $\lambda_2 \leqslant \lambda_1$. Inequality (2.4.21) gives no condition on the rates and condition $\lambda_2 \leqslant \lambda_1$ is sufficient to construct the coupled process $(\widetilde{\xi}_t, \eta_t)_{t\geqslant 0}$ via the following dynamics :

transition	rate
$(3,1) \longrightarrow (2,0)$	1
$ (3,3)$	r
$(2,0) \longrightarrow (2,2)$	r
$ (3,1)$	$\lambda_2 n_3(x,\widetilde{\xi})$
$ (2,1)$	$\lambda_1 n_1(x,\eta) + \lambda_2(n_3(x,\eta) - n_3(x,\widetilde{\xi}))$
$(2,2) \longrightarrow (3,3)$	$\lambda_2 n_3(x,\widetilde{\xi})$
$ (2,3)$	$\lambda_1 n_1(x,\eta) + \lambda_2(n_3(x,\eta) - n_3(x,\widetilde{\xi}))$
$ (2,0)$	1
$(3,3) \longrightarrow (2,2)$	1
$ (3,1)$	1
$(2,3) \longrightarrow (2,2)$	1
$ (2,1)$	1
$ (3,3)$	$\lambda_2 n_3(x,\widetilde{\xi})$
$(2,1) \longrightarrow (3,1)$	$\lambda_2 n_3(x,\widetilde{\xi})$
$ (2,3)$	r
$ (2,0)$	1
$(3,2) \longrightarrow (2,0)$	1
$ (3,3)$	$\lambda_1 n_1(x,\eta) + \lambda_2 n_3(x,\eta)$
$(3,0) \longrightarrow (3,1)$	$\lambda_1 n_1(x,\eta) + \lambda_2 n_3(x,\eta)$
$ (3,2)$	r
$ (2,0)$	1

whose rate $\lambda_1 n_1(x,\eta) + \lambda_2(n_3(x,\eta) - n_3(x,\widetilde{\xi}))$ is well defined : since

$$n_3(x,\widetilde{\xi}) = |\{y \sim x : \widetilde{\xi}(y) = 3, \eta(y) = 1\}| + |\{y \sim x : \widetilde{\xi}(y) = 3, \eta(y) = 3\}|$$

$$n_1(x,\eta) = |\{y \sim x : \widetilde{\xi}(y) = 3, \eta(y) = 1\}| + |\{y \sim x : \widetilde{\xi}(y) = 2, \eta(y) = 1\}|$$

$$n_3(x,\eta) = |\{y \sim x : \widetilde{\xi}(y) = 3, \eta(y) = 3\}| + |\{y \sim x : \widetilde{\xi}(y) = 2, \eta(y) = 3\}|$$

along with $\lambda_2 \leqslant \lambda_1$, gives

$$\begin{aligned} \lambda_1 n_1(x,\eta) + \lambda_2(n_3(x,\eta) - n_3(x,\widetilde{\xi})) &= (\lambda_1 - \lambda_2)|\{y \sim x : \widetilde{\xi}(y) = 3, \eta(y) = 1\}| \\ &+ \lambda_2(|\{y \sim x : \widetilde{\xi}(y) = 2, \eta(y) = 3\}| + \lambda_1|\{y \sim x : \widetilde{\xi}(y) = 2, \eta(y) = 1\}|) \end{aligned}$$

$\square$

2.5 Phase transition

In this section, we take advantage of all the stochastic order relations between processes established in Section 2.4 to derive results for a phase transition of the multi-type process $(\eta_t)_{t\geqslant 0}$, in both symmetric and asymmetric cases. According to Definition

2.2.1, we assume the multitype process to have initial configuration $\eta_0 = \mathbf{1}_{\{0\}}$ and note $\eta_t = \eta_t^{\{0\}}$.

As announced in Section 2.2, we first deal with the cases where $\lambda_2 < \lambda_1$ are both smaller or larger than λ_c.

Proof of Proposition 2.2.1. Let $(\xi_t)_{t\geqslant 0}$ be a basic contact process with growth rate λ_1 and let $(\eta_t)_{t\geqslant 0}$ be a symmetric multitype process with parameters $(\lambda_1, \lambda_2, r)$ such that $\eta_0 \leqslant \xi_0$. By Proposition 2.4.5, $(\xi_t)_{t\geqslant 0}$ is stochastically larger than $(\eta_t)_{t\geqslant 0}$. Since $\lambda_1 \leqslant \lambda_c$, $(\xi_t)_{t\geqslant 0}$ is subcritical, thus, the symmetric multitype process dies out.

The extinction of the asymmetric multitype process is a consequence of the extinction of the symmetric process and Proposition 2.4.4. □

Proof of Proposition 2.2.2. Let $(\widetilde{\xi}_t)_{t\geqslant 0}$ be a basic contact process with growth rate λ_2 on $\{2,3\}^{\mathbb{Z}^d}$ and let $(\eta_t)_{t\geqslant 0}$ be a symmetric multitype process with parameters $(\lambda_1, \lambda_2, r)$. By Proposition 2.4.6, $(\xi_t)_{t\geqslant 0}$ is stochastically lower than $(\eta_t)_{t\geqslant 0}$. Since $\lambda_2 > \lambda_c$, the process $(\widetilde{\xi}_t)_{t\geqslant 0}$ is supercritical and therefore, the symmetric multitype process survives. □

We now turn to Theorems 2.2.1 and 2.2.2, for which we shall prove :

Theorem 2.5.1. *Assume $\lambda_2 < \lambda_c < \lambda_1$ fixed. Let $(\eta_t)_{t\geqslant 0}$ be the symmetric multitype process. Then,*

(i) there exists $r_0 \in (0,\infty)$ such that if $r < r_0$ then the process $(\eta_t)_{t\geqslant 0}$ survives.

(ii) there exists $r_1 \in (0,\infty)$ such that if $r > r_1$ then the process $(\eta_t)_{t\geqslant 0}$ dies out.

Theorem 2.5.2. *Assume $\lambda_c < \lambda_1$ and $\lambda_2 < \lambda_1$ fixed. Let $(\eta_t)_{t\geqslant 0}$ be the asymmetric multitype process. Then,*

(i) there exists $s_0 \in (0,\infty)$ such that if $r < s_0$ then the process $(\eta_t)_{t\geqslant 0}$ survives.

(ii) there exists $s_1 \in (0,\infty)$ such that if $r > s_1$ then the process $(\eta_t)_{t\geqslant 0}$ dies out.

These results imply Theorems 2.2.1 and 2.2.2, that is, the existence of a unique phase transition with a critical value r_c (resp. s_c) defined in (2.2.12). Indeed, relying on Theorems 2.5.1 and 2.5.2, by monotonicity given by Corollary 2.4.2 one has $r_0 = r_1$ (resp. $s_0 = s_1$).

We shall prove both theorems in Subsections 2.5.2 and 2.5.3. One concludes for the critical case by proving Theorem 2.2.3 in Subsection 2.6.

Before that, subsection 2.5.1 deals with consequences of Theorems 2.5.1 and 2.5.2 along with monotonicity results of Section 2.4.

2.5.1 Behaviour of the critical value with varying growth rates

Suppose the existence of the critical value r_c guaranteed in virtue of Theorems 2.5.1 and 2.5.2, one investigates the behaviour of r_c when growth rates λ_1 and λ_2 are moving. One manages to prove monotonicity between growth rates and the release rate, in the sense that, the more competitive the species is (i.e. the higher the parameter λ_2 is) or the fittest the species is (i.e. the higher the parameter λ_1 is), the higher the release rate is (i.e. the higher the critical value r_c is) :

Proposition 2.5.1. *For $j = 1, 2$, the function $\lambda_j \longmapsto r_c(\lambda_j)$ is non-increasing.*

Proof. We consider $j = 2$ as the case $j = 1$ is similar. Let $(\eta_t)_{t\geqslant 0}$ and $(\eta'_t)_{t\geqslant 0}$ be two multitype processes with respective parameters $(\lambda_1, \lambda_2, r)$ and $(\lambda_1, \lambda'_2, r)$. By Theorems 2.5.1 and 2.5.2, existence and uniqueness of the critical values r_c and r'_c associated to those processes are guaranteed. We now show that if $\lambda_2 < \lambda'_2$, then $r_c \leqslant r'_c$.

By contradiction, suppose $r_c > r'_c$. Let r be fixed be such that $r_c > r > r'_c$. Since $\lambda_2 < \lambda'_2$, by Corollary 2.4.1-(iii), if $\eta_0 = \eta'_0$ then $\eta_t \leqslant \eta'_t$ a.s. By Theorem 2.2.3 and Corollary 2.4.1,

$$\mathbb{P}_r(\forall t \geqslant 0,\ H'_t \neq \varnothing) \leqslant \mathbb{P}_{r'_c}(\forall t \geqslant 0,\ H'_t \neq \varnothing) = 0$$

But since $r < r_c$, the process $(\eta_t)_{t\geqslant 0}$ survives : $\mathbb{P}_r(\forall t \geqslant 0,\ H_t \neq \varnothing) > 0$. This contradicts $\eta_t \leqslant \eta'_t$ a.s., hence $r_c \leqslant r'_c$. □

2.5.2 Subcritical case

The following proof relies on a comparison of the multitype process with an oriented percolation process on the even grid $\mathcal{L}$. Then we show that for the associated open sites, percolation occurs thanks to results we presented in Section 2.2.3.

We follow arguments used by N. Konno, R. Schinazi and H. Tanemura [48] in the case of a spatial epidemic model.

Proof of Theorem 2.5.1 (i). To simplify notations, choose $d = 1$ but the proof remains the same for any $d \geqslant 2$. Introduce the following space-time regions,

$$\begin{aligned} &\mathcal{B} = (-4L, 4L) \times [0, T], \quad \mathcal{B}_{m,n} = (2mLe_1, nT) + \mathcal{B} \\ &I = [-L, L], \quad I_m = 2mLe_1 + I \end{aligned}$$

for positive integers L, T to be chosen later, where $(e_1, ..., e_d)$ denotes the canonical basis of $\mathbb{R}^d$. Notice they correspond to the boxes introduced in (2.2.20) with $j_0 = 1$, $k_0 = 4$.

Consider the process $(\eta_t^{m,n})_{t\geqslant 0}$ restricted to the region $\mathcal{B}_{m,n}$, that is, constructed from the graphical representation where only arrival times of the Poisson processes occurring within $\mathcal{B}_{m,n}$ are taken into account. Therefore, a birth on a site $x \in \mathcal{B}_{m,n}$ from some site y only occurs if $y \in \mathcal{B}_{m,n}$. By Proposition 2.4.2 and Remark 2.4.1, one has

$$\eta_t^{m,n} \leqslant \eta_t\big|_{\mathcal{B}_{m,n}}, \tag{2.5.1}$$

for all $t > 0$ if $\eta_0^{m,n} = \eta_0\big|_{\mathcal{B}_{m,n}}$.

Let $k = \lfloor\sqrt{L}\rfloor$ and define $\mathcal{C} = [-k, k]$. One declares $(m, n) \in \mathcal{L}$ to be *wet* if for any configuration at time nT such that there is a translate of $\mathcal{C}$ full with 1's in I_m with I_m

containing only 0's and 1's, the process restricted to $\mathcal{B}_{m,n}$ is such that at time $(n+1)T$ there are a translate of $\mathcal{C}$ in I_{m-1} and a translate of $\mathcal{C}$ in I_{m+1}, both full of 1's, with I_{m-1} and I_{m+1} containing only 0's and 1's.

Let us show that the probability of a site $(m,n) \in \mathcal{L}$ to be wet can be made arbitrarily close to 1 for L and T chosen sufficiently large. By translation invariance, it is enough to deal with the case $(m,n) = (0,0)$.

Suppose I contains only 0's and 1's as well as the translate of $\mathcal{C}$ full of 1's and set $r = 0$ in $\mathcal{B}$, that is, no more type-2 individuals arrive in the box $\mathcal{B}$ after time 0.

If type-2 individuals are present on the base $(-4L,-L)\cup(L,4L)\times\{0\}$, the probability of the event E they all die by time $T/2$ is at least

$$\Big(1-\exp(-T/2)\Big)^{6L}$$

which is larger than $1-\epsilon$ for T and L chosen large enough. On E, the process restricted to the box $\mathcal{B}$ is now from time $T/2$ a supercritical contact process $(\xi_t^{m,n})_{t\geqslant T/2}$ with distribution $\widetilde{\mathbb{P}}(\xi_t^{m,n} \in \cdot)$. But we have to make sure that at time $T/2$, there are still enough 1's for $\xi_{T/2}^{m,n}$, for this we use the following result. Define $\tau(\ell) = \inf\{t > 0 : \Xi_t^{[-\ell,\ell]}\Big|_{[-\ell,\ell]} = \varnothing\}$, the hitting time of the trap state $\underline{0}$ of the contact process starting from $[-\ell,\ell]$ and restricted to $[-\ell,\ell]\times[0,T/2]$. T. Mountford [62] proved that

$$\widetilde{\mathbb{P}}(\tau(\ell) \leqslant \exp(\ell)) \leqslant \exp(-\ell) \ \text{ for } \ell \text{ large enough} \tag{2.5.2}$$

Partition $\mathcal{C}$ into $M = \lfloor\sqrt{k}\rfloor$ boxes, each of them being a translate of $[0,M]$. From each of these M boxes, say box $j \leqslant M$ run a supercritical contact process denoted by $(\zeta_t^j)_{t\geqslant 0}$ which coincides with the restriction of $\xi_t^{m,n}$ to this box. Therefore for each x in this box J, as in 2.5.1, $\zeta_{T/2}^j(x) \leqslant \xi_{T/2}^{m,n}(x)$ for all $x \in \mathcal{B}_{m,n}$. Then for the union of these j boxes $(j \leqslant M)$, the probability there is at least M 1's within $\mathcal{C}$ by time $T/2$ is after (2.5.2), with T such that $\exp(M) \leqslant T/2$, at least

$$\widetilde{\mathbb{P}}(\tau(M) \geqslant T/2)^M \geqslant \widetilde{\mathbb{P}}(\tau(M) \geqslant \exp(M))^M \geqslant (1-\exp(-M))^M \tag{2.5.3}$$

which can be made larger than $1-\epsilon$, for M, i.e. L, large enough.

A result of R. Durrett and R. Schinazi [25] shows that for a contact process $(\xi_t)_{t\geqslant 0}$, for any $A \subset \mathbb{Z}$, except for a set with exponentially small probability, either $\Xi_t^A = \varnothing$, or $\xi_t^A = \xi_t^{\mathbb{Z}}$ on a linearly time growing set $[-\alpha t, \alpha t]$: there exists $\alpha > 0$ such that for all $A \subset \mathbb{Z}$, there exist positive constants C, γ such that

$$\widetilde{\mathbb{P}}(\Xi_t^A \neq \varnothing,\ \xi_t^A(x) \neq \xi_t^{\mathbb{Z}}(x)) \leqslant C\exp(-\gamma t) \tag{2.5.4}$$

where $x \in A + \alpha t$.

We applied this result with $A \subset \mathcal{C}$ which correspond to the numbers of 1's in the box. We just proved that $|A| > |M|$. Moreover according to Proposition 2.6.1, one can choose k, and so L, large enough so that this supercritical contact process ξ_t^A starting from at least M 1's survives at time $T/2$ with probability close to 1, hence $\xi_t^A \neq \emptyset$ and (2.5.4) is valid. In this situation, taking $T/2 = 9L/(2\alpha)$ with L large enough, the process $(\xi_t^{m,n})_{t\geqslant 0}$ starting from at least M 1's in $[-L, L]$ at time $T/2$ will be coupled with a process $\xi_t^{\mathbb{Z}}$ on $[-3L, 3L]$ with probability at least $1-\epsilon$ at time T. Hence, since $3L > \alpha T/2 > 2L$, by time T the contact process ξ_t^A started inside $[-L, L]$ has not reached the boundary of $[-4L, 4L]$ with probability close to 1. Then, the process $\xi_t^{m,n}$ and the contact process ξ_t^A are the same with probability $1-\epsilon$ in $[-4L, 4L]$; this way, the coupling of $(\xi_t^{m,n})_{t\geqslant 0}$ with $(\xi_t^{\mathbb{Z}})_{t\geqslant 0}$ works so far with probability $1-\epsilon$ if L is large enough.

Since the distribution of $(\xi_t^{\mathbb{Z}})_{t\geqslant 0}$ is stochastically larger than the upper invariant measure $\overline{\nu}$ (see Chapter 1 Section 1.1.2) of the contact process, on the survival event, $\overline{\nu}$ loads a positive density ρ of 1's. Since $\overline{\nu}$ is ergodic (see Chapter 1 Section 1.1.2),

$$\lim_{L\to\infty} \frac{1}{2L+1} \sum_{x=-3L}^{-L} \mathbf{1}\{\eta(x) = 1\} = \rho \quad \overline{\nu}\text{ -a.e.}$$

In other words, as soon as L is large enough, under $\overline{\nu}$ there are at least k 1's in any interval of length $2L+1$ with $\overline{\nu}$-probability at least $1-\epsilon$. Since we obtained that $(\xi_t^{m,n})_{t\geqslant 0}$ is coupled to $(\xi_t^{\mathbb{Z}})_{t\geqslant 0}$ by time T with probability at least $1-2\epsilon$, for L large enough, there are at least k 1's in $[-3L, -L]$ at time T with probability at least $1-2\epsilon$ and similarly, at least k 1's at time t in $[L, 3L]$ with probability at least $1-2\epsilon$ as well for $(\xi_t^{m,n})_{t\geqslant 0}$. Consequently,

$$\widetilde{\mathbb{P}}((0,0) \text{ wet}) > 1 - 4\epsilon, \text{ if } r = 0. \tag{2.5.5}$$

Since $\mathcal{B}$ is a finite space-time region, for fixed L, T, one can pick $r_0 > 0$ small enough so that the arrival times of a rate r Poisson process, such that $r < r_0$, in $\mathcal{B}$ occurs with probability at most ϵ. Let $A_{L,T}(r)$ be the first arrival time of a rate r Poisson process in $[-2L, 2L] \times [0, T]$.

$$\begin{aligned}
\mathbb{P}_r((0,0) \text{ wet}) &\geqslant \mathbb{P}_r((0,0) \text{ wet}, A_{L,T}(r) > T)\mathbb{P}_r(A_{L,T}(r) > T) \\
&\geqslant (1-4\epsilon)e^{-r(4L+1)T} \\
&\geqslant 1 - 6\epsilon
\end{aligned}$$

as soon as the exponent of the exponential is close to 0, i.e. by choosing r small enough.

See Figure 2.2 for an illustration.

Now construct a percolation process by defining the *good event* $G_{m,n} = \{(m,n) \text{ wet}\}$. Notice that $G_{m,n}$ depends only on the process constructed in $\mathcal{B}_{m,n}$, and for $(a,b) \in \mathcal{L}$, events $G_{m,n}$ and $G_{a,b}$ are independent if (m,n) and (a,b) are not neighbours. The events $\{G_{m,n}, (m,n) \in \mathcal{L}\}$ are thus 1-dependent. By the comparison theorem 2.2.7, the process $(\eta_t^{m,n})_{t\geqslant 0}$ restricted to regions $\mathcal{B}_{m,n}$ is stochastically larger than a 1-dependent percolation process with probability $1-\epsilon$.

By Lemma 2.2.5, one can choose ϵ small enough so that percolation occurs in the 1-dependent percolation process with density $1-\epsilon$. □

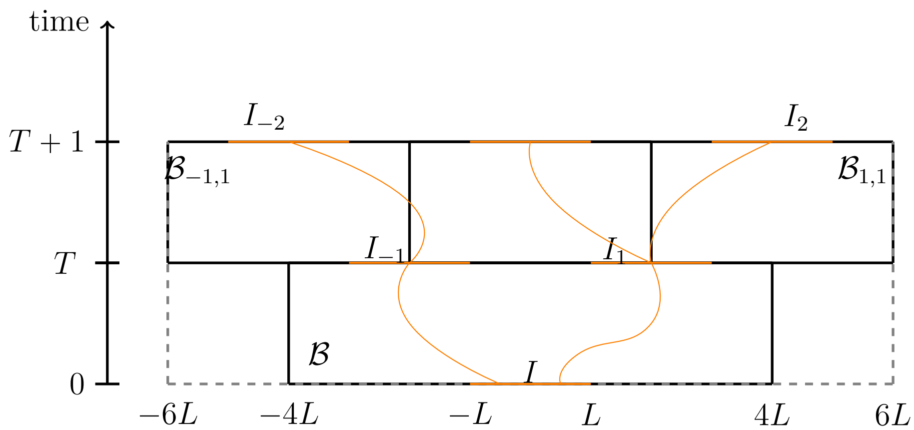

FIGURE 2.2: There exist L, T such that $(0,0)$ is wet with $\mathbb{P}_r$-probability close to 1.

2.5.3 Supercritical case

In the following, one compares our particle system with a percolation process on $\mathbb{Z}^2 \times \mathbb{Z}_+$ and uses arguments from Van Den Berg et al. [75].

Proof of Theorem 2.2.1 (ii). Assume $d = 2$, the proof can similarly be extended to higher dimensions. For all $(k, m, n) \in \mathbb{Z}^2 \times \mathbb{Z}_+$. Introduce the following space-time regions, for positive L and T to be chosen later.

$$\mathcal{A} = [-2L, 2L]^2 \times [0, 2T] \qquad \mathcal{A}_{k,m,n} = \mathcal{A} + (kL, mL, nT)$$
$$\mathcal{B} = [-L, L]^2 \times [T, 2T] \qquad \mathcal{B}_{k,m,n} = \mathcal{B} + (kL, mL, nT)$$
$$\mathcal{C} = \mathcal{C}_{bottom} \bigcup \mathcal{C}_{side} \qquad \mathcal{C}_{k,m,n} = \mathcal{C} + (kL, mL, nT)$$

where $\mathcal{C}_{bottom} = \{(m, n, t) \in \mathcal{A} : t = 0\}$
$\mathcal{C}_{side} = \{(m, n, t) \in \mathcal{A} : |m| = 2L \text{ or } |n| = 2L\}$

Consider a restriction of the process $(\eta_t)_{t\geqslant 0}$ to $\mathcal{A}_{k,m,n}$, that is, the process $(\eta_t^{k,m,n})_{t\geqslant 0}$ constructed from its graphical representation within $\mathcal{A}_{k,m,n}$.

One declares a site $(k, m, n) \in \mathbb{Z}^2 \times \mathbb{Z}_+$ to be *wet* if the process $(\eta_t^{k,m,n})_{t\geqslant 0}$ contains no wild individual in $\mathcal{B}_{k,m,n}$ starting from any configuration in $\mathcal{C}_{k,m,n}$. Therefore it will be the same for $\eta_t\big|_{\mathcal{A}_{k,m,n}}$. Sites that are not wet are called *dry*.

For any $\epsilon > 0$, we show that for some chosen L and T any site (k, m, n) is wet with probability close to 1 when r is large enough. By translation invariance, it is enough to consider $(k, m, n) = (0, 0, 0)$. Set $r = \infty$ in $\mathcal{A}$. Then, the process $(\eta_t^{k,m,n})_{t\geqslant 0}$ contains only sites in state 2 or 3 : sites in state 0 or 1 flip instantaneously into state 2 and 3 respectively. That is, $(\eta_t^{k,m,n})_{t\geqslant 0}$ is in fact a contact process $(\widetilde{\xi}_t^{k,m,n})_{t\geqslant 0}$ on $\{2,3\}^{[-2L,2L]}$ The contact process $(\widetilde{\xi}_t)_{t\geqslant 0}$ on $\{2,3\}^{\mathbb{Z}^2}$ with growth rate $\lambda_2 < \lambda_c$ is subcritical.

If there is some wild individual in $\mathcal{B}$, it should have come from a succession of births started somewhere in $\mathcal{C}$. Starting from a site in $\mathcal{C}_{side}$, a path to $\mathcal{B}$ should last at least L

sites; according to C. Bezuidenhout and G. Grimmett [6] there exists such a path with probability at most $C\exp(-\gamma L)$, for some positive constants C, γ. Hence,

$$\mathbb{P}(\exists(x,t) \in \mathcal{C}_{side} \times [0,2T] : (x,t) \to \mathcal{B}) \leqslant 4\big(2T \times (4L+1)\big)C\exp(-\gamma L)$$

Similarly, starting from the base $\mathcal{C}_{bottom}$, there exists a path lasting at least T sites with probability

$$\mathbb{P}(\exists(x,t) \in \mathcal{C}_{bottom} : (x,t) \to \mathcal{B}) \leqslant (4L+1)^2 C\exp(-\gamma T)$$

Consequently if $r = \infty$,

$$\mathbb{P}((0,0,0) \text{ wet }) \geqslant 1 - 4\big(2L \times (4L+1)\big)Ce^{-\gamma L} - (4L+1)^2 Ce^{-\gamma T} \geqslant 1 - \epsilon/2,$$

for L and T large enough.

Since $\mathcal{A}$ is a finite space-time region, one can pick r large enough so that with probability at least $1-\epsilon/2$, an exponential clock with parameters r rings before any other so that there are no type-1 individuals in $\mathcal{A}$ with probability close to 1 :

$$\mathbb{P}_r(k,m,n) \text{ wet}) \geqslant 1 - \epsilon$$

for r large enough.

To construct a percolation process on $\mathbb{Z}^2 \times \mathbb{Z}_+$, one puts an oriented arrow from (k,m,n) to (x,y,z) if $n \leqslant z$ and if $\mathcal{A}_{k,m,n} \cap \mathcal{A}_{x,y,z} \neq \varnothing$. The event $G_{k,m,n} = \{(k,m,n) \text{ wet}\}$ depends only on the graphical construction of the process within $\mathcal{A}_{k,m,n}$, furthermore, for all $(k,m,n) \in \mathbb{Z}^2 \times \mathbb{Z}_+$, there is a finite number of sites $(x,y,z) \in \mathbb{Z}^2 \times \mathbb{Z}_+$ such that $\mathcal{A}_{k,m,n} \cap \mathcal{A}_{x,y,z} \neq \varnothing$. The percolation process is dependent but of finite range. The existence of a path of wild individuals for the particle system corresponds to a path of dry sites for the percolation and we proved that dry paths are unlikely.

Let us show that for all sites, there exists a finite random time after which there is no more wild individuals remaining. Follow the construction given by van den Berg et al. [75].

Since the percolation is of finite range, there exists some positive constant $C(d)$ such that if the distance between two sites is at least $C(d)$ then they are mutually independent. For any $x \in \mathbb{Z}^2$, define $T_x = \sup\{t : \eta_t(x) \in \{1,3\}\}$ the last time where x is occupied by a wild individual. By translation invariance, deal with the case $x = (0,0)$.

Let $K > 0$, suppose $T_0 > TK$, there exists some $m \in \mathbb{Z}_+$ such that $(0,0,m)$ is the end of a dry path starting from $(x,y,0)$ with $(x,y) \in \mathbb{Z}^2$. The number of paths of length ℓ is at most δ^ℓ. Moreover, a self-avoiding path of length ℓ contains at most $\nu\ell$ mutually independent sites (i.e. whose in-between distance is at least $C(d)$), $\nu > 0$. Hence,

$$\mathbb{P}(T_0 > TK) \leqslant \sum_{m \geqslant K-1} \sum_{\ell \geqslant m} \delta^\ell \big(1 - \mathbb{P}((k,m,n) \text{ wet })\big)^{\nu\ell} \tag{2.5.6}$$

For r large enough,the right-hand side tends to 0 when K goes to infinity. That is, T_0 is almost surely finite and the region $\mathcal{A} \times [T_x, \infty)$ is wild individuals-free. For the percolation process, this means there is an infinite path of wet sites, hence the process $(\eta_t)_{t\geqslant 0}$ dies out. □

To sum up, we just showed there exist r_0 and r_1 such that $r_0 \leqslant r_c \leqslant r_1$, for $r \leqslant r_0$ the process survives and for $r \geqslant r_1$ the process dies out.

This proves the existence of a phase transition for the symmetric multitype process. The proof of Theorem 2.5.1-(i) only uses that contact process with growth rate λ_1 is supercritical, this is also true to show the existence of s_0 in Theorem 2.5.2-(i). By Proposition 2.4.4, the asymmetric multitype process dies out as soon as the symmetric one does, existence of s_1 in Theorem 2.5.2-(ii) is then immediately guaranteed by Theorem 2.5.1-(ii). Though, one can remark that conditions of Theorem 2.5.2 are milder : one can actually show the existence of s_1 in a neater way. Indeed, retrieving briefly the proof of the supercritical case : assume $\lambda_2 > \lambda_c$: with the lack of the transition "$2 \to 3$" in the asymmetric case and choosing first $r = \infty$, one notices for the subcritical contact process on $\{2,3\}^{\mathbb{Z}^2}$, there are no possible paths of wild individuals created by the 3's from the boundary $\mathcal{C}_{k,m,n}$ up to extinction, but this occurs exponentially fast (see C. Bezuidenhout and G. Grimmett [6]).

2.6 The critical process dies out

In this section, we prove Theorem 2.2.3 : the critical multi-type contact process dies out. i.e. $\mathbb{P}_{r_c}(H_t \neq \varnothing \ \forall t \geqslant 0) = 0$. Recall $(\eta_t)_{t\geqslant 0}$ stands for the multitype process, starting from the initial configuration $\eta_0 = \mathbf{1}_{\{0\}}$.

One follows closely the arguments used by C. Bezuidenhout and G. Grimmett [5], well-exposed by T.M. Liggett [57, Chapter I.2]. We shall use both presentations.

The *dynamic renormalization* construction sees the time-evolution of the process in a suitable chosen scaling : space-time is divided into *finite* space-time regions. So far, this looks heavily like the comparison with oriented percolation we defined in Section 2.2. But here, instead of fixing every region initially, the idea is rather to determine their positions according to the past random position in the construction, along with the evolution of the process.

Let us sketch the contents of the proof.

Outline. The first step consists to observe that if the process survives in an arbitrary large box, then it reaches its boundaries densely. We shall estimate these densities at each side of a space-time region.

This way, one can repeat this step by running the process in an other adjacent box starting from the boundary of the previous one and so on, conditionally on the fact that the starting configuration is dense enough. This is the second step. In connection with the proofs of Theorems 2.5.2 and 2.5.1 where we looked after having translations of occupied finite intervals at a given fixed time, here we look after having translations of the densities in some space-time slab.

Now, knowing that at each stage, one can construct overall a path of adjacent boxes wherein the process survives and reaches the boundaries densely, it remains to compare the process with an oriented percolation process to extend the arguments to infinite space and time. As before, compare a space-time box to a vertex in the even lattice of an oriented percolation so that one declare a vertex to be wet if some good event associated to the box is a success. Conclude thanks to results about percolation theory, recalled in Section 2.2.

2.6.1 Local characterization of the survival event

We saw under specific hypothesis on r, the multi-type contact process survives with positive probability. Supposing it survives, one exhibits here several properties of growth satisfied by the process restricted to an arbitrary large box. Such results have been proved for the basic contact process by C. Bezuidenhout and G. Grimmett [5], thanks to techniques of dynamic renormalization introduced by G. Grimmett et al. [35, 2].

First note the arguments developed by [5] rely on elementary properties of the contact process making them robust. They are also valid for the multitype process because the latter satisfies the following properties we have exhibited in previous sections :

(A) contact process-like dynamics : one retrieves the growth rate λ_1 or λ_2 of a basic contact process, even if it is determined randomly. We will make use of the more suitable one depending on the situation.

(B) attractiveness, by Section 2.4.

(C) correlation inequalities : using correlation inequalities such as FKG inequality 1.2.1.

Note that the use of (C) is possible because we shall work in finite space-time regions in the following. Such techniques have been several times exploited to study critical processes, including works by O. Garet and R. Marchand [30] for a branching random walk, J. Steif and M. Warfheimer [74] for a randomly evolving contact process.

Covering of an arbitrary large box

Proposition 2.6.1. *Suppose $(\eta_t)_{t\geqslant 0}$ survives, then*

$$\lim_{n\to\infty} \mathbb{P}_r(H_t^{[-n,n]^d} \neq \varnothing \ \forall t) = 1. \tag{2.6.1}$$

Proof. By attractiveness of the process $(\eta_t)_{t\geqslant 0}$ (see Corollary 2.4.1), if $A, B \subset \mathbb{Z}^d$ are such that $A \subset B$ then

$$\mathbb{P}_r(H_t^B \neq \varnothing \ \forall t \geqslant 0) \geqslant \mathbb{P}_r(H_t^A \neq \varnothing \ \forall t \geqslant 0) > 0.$$

Since we assumed $(\eta_t)_{t\geqslant 0}$ survives, $\lim_{A\uparrow\mathbb{Z}^d} \mathbb{P}_r(H_t^A \neq \varnothing \ \forall t) = 1.$ □

Consider, for $L \geqslant 1$ and $A \subset \mathbb{Z}^d$, the truncated process $({}_L\eta_t^A)_{t\geqslant 0}$ defined as the process $(\eta_t)_{t\geqslant 0}$ starting from the initial configuration $\eta_0 = \mathbf{1}_A$ constructed from the graphical representation in $(-L, L)^d \times [0, t]$. Denote by $({}_L H_t^A)_{t\geqslant 0}$ the associated set of sites occupied by wild individuals at time t.

The next two results show that sites occupied by wild individuals are dense in some orthant of the top of a box of size $(-L, L)^d \times [0, T]$. Following estimates are analogous to the ones we did previously in the proof of Theorem 2.5.1, Subsection 2.5.2, one proves by (A) and (B) of $(\eta_t)_{t\geqslant 0}$ that

Proposition 2.6.2. *Let $n \geqslant 1$ and $N \geqslant 1$, then*

$$\lim_{t\to\infty} \lim_{L\to\infty} \mathbb{P}_r(|{}_L H_t^{[-n,n]^d}| \geqslant N) = \mathbb{P}_r(H_t^{[-n,n]^d} \neq \varnothing \ \forall t > 0) \tag{2.6.2}$$

Proof. Since $H_t^{[-n,n]^d} = \bigcup_{L\geqslant 0} {}_L H_t{}^{[-n,n]^d}$, for any fixed t, by monotonicity (see Corollary 2.4.1),

$$\lim_{L\to\infty} \mathbb{P}_r(|{}_L H_t^{[-n,n]^d}| \geqslant N) = \mathbb{P}_r(|H_t^{[-n,n]^d}| \geqslant N). \tag{2.6.3}$$

It is thus enough to show

$$\lim_{t\to\infty} \mathbb{P}_r(|H_t^{[-n,n]^d}| \geqslant N) = \mathbb{P}_r(H_t^{[-n,n]^d} \neq \varnothing \ \forall t > 0).$$

On the other hand, for an initial configuration constituted of $(2n+1)^d$ wild individuals, the probability that these $(2n+1)^d$ wild individuals die before any birth is at least the probability the maximum of $(2n+1)^d$ independent exponential clocks with parameter 1 is smaller than the minimum of $2d(2n+1)^d$ independent exponential clocks with parameter λ_2, since $\lambda_2 < \lambda_1$. That is,

$$\mathbb{P}_r(H_t^{[-n,n]^d} = \varnothing \text{ for some } t|\mathcal{F}_s) \geqslant \left[\frac{1}{1 + 2d\lambda_2|H_s^{[-n,n]^d}|}\right]^{|H_s^{[-n,n]^d}|}$$

where $\mathcal{F}_t = \sigma(\eta_s^{[-n,n]^d},\ s \leqslant t)$ is the sigma-algebra generated by the graphical representation of the process $(\eta_t^{[-n,n]^d})_{t\geqslant 0}$ until time t.

Define $\mathcal{F}_\infty = \bigcap_{s\geqslant 0} \sigma(\mathcal{F}_s)$, since $\{H_t^{[-n,n]^d} = \varnothing \text{ for some } t\}$ is a tail-event with respect to $\mathcal{F}_\infty$, and $\mathbf{1}\{H_t^{[-n,n]^d} = \varnothing \text{ for some } t\}$ is $\mathbb{P}_r$-integrable, by the martingale convergence theorem, Lévy's zero–one law gives

$$\lim_{s\to\infty} \mathbb{E}[\mathbf{1}\{H_t^{[-n,n]^d} = \varnothing \text{ for some } t\}|\mathcal{F}_s] = \mathbf{1}\{H_t^{[-n,n]^d} = \varnothing \text{ for some } t\} \ \text{ a.s.}$$

Therefore,

$$\lim_{t\to\infty} |H_t^{[-n,n]^d}| = \infty \text{ a.s. on} \{H_s^{[-n,n]^d} \neq \varnothing \ \forall s \geqslant 0\}, \tag{2.6.4}$$

and by (2.6.3) and (2.6.4),

$$\lim_{t\to\infty} \mathbb{P}_r(|H_t^{[-n,n]^d}| \geqslant N, H_s^{[-n,n]^d} \neq \varnothing\ \forall s > 0) = \mathbb{P}_r(H_t^{[-n,n]^d} \neq \varnothing\ \forall t \geqslant 0)$$

□

Using FKG inequality (C), one shows that the truncated process contains a large number of occupied sites in some orthant of $\mathbb{R}^d$. For this, define the 2^d orthants of $\mathbb{R}^d$: for $u = (u_1, ..., u_d) \in \{-,+\}^d$,

$$\mathcal{O}^u := \{x = (x_1, ..., x_d) \in \mathbb{R}^d :\ sgn(x_i) = u_i,\ 1 \leqslant i \leqslant d\}.$$

By symmetry and reflexion with respect to the time axis, without loss of generality one can only consider the positive orthant i.e. when $sgn(x_i) = +$ for any $1 \leqslant i \leqslant d$ that we denote by

$$\mathcal{O}^+ := \{x = (x_1, ..., x_d) \in \mathbb{R}^d :\ sgn(x_i) = +,\ 1 \leqslant i \leqslant d\}.$$

Proposition 2.6.3. *Fix $n \geqslant 1$, $N \geqslant 1$ and $L \geqslant n$,*

$$\mathbb{P}_r(|{}_LH_t^{[-n,n]^d} \cap \mathcal{O}^+| \leqslant N)^{2^d} \leqslant \mathbb{P}_r(|{}_LH_t^{[-n,n]^d}| \leqslant 2^d N) \tag{2.6.5}$$

Proof. Along this proof, make us of (A) since we retrieve a basic contact process, so that one use the suitable growth rate depending on the ongoing estimate. First, remark that

$$|{}_LH_t^{[-n,n]^d}| = \sum_{u\in\{-,+\}^d} |{}_LH_t^{[-n,n]^d} \cap \mathcal{O}^u|.$$

All $\{|{}_LH_t^{[-n,n]^d} \cap \mathcal{O}^u|, u \in \{-,+\}^d\}$ are independent, identically distributed and positively correlated by monotonicity : increasing with respect to growth rate Poisson process and decreasing with respect to death and release rates Poisson processes, using (B) by Corollary 2.4.1. So that by FKG inequality, for all $u \in \{-,+\}^d$:

$$\begin{aligned}
\left(\mathbb{P}_r\left(|{}_LH_t^{[-n,n]^d} \cap \mathcal{O}^+| \leqslant N\right)\right)^{2^d} &= \prod_{u\in\{-,+\}^d} \mathbb{P}_r\left(|{}_LH_t^{[-n,n]^d} \cap \mathcal{O}^u| \leqslant N\right)\\
&\leqslant \mathbb{P}_r\left(\bigcap_{u\in\{-,+\}^d} \left(|{}_LH_t^{[-n,n]^d} \cap \mathcal{O}^u| \leqslant N\right)\right)\\
&\leqslant \mathbb{P}_r\left(\sum_{u\in\{-,+\}^d} |{}_LH_t{}^{[-n,n]^d} \cap \mathcal{O}^u| \leqslant 2^d N\right)\\
&\leqslant \mathbb{P}_r\left(|{}_LH_t^{[-n,n]^d}| \leqslant 2^d N\right).
\end{aligned}$$

□

By Propositions 2.6.2 and 2.6.3, for any $\epsilon > 0$, there exist L and t sufficiently large such that

$$\mathbb{P}_r(|_L H_t^{[-n,n]^d} \cap \mathcal{O}_d^u| \geqslant N) > 1 - \epsilon^{2^d}.$$

Before going on space-time conditions, consider the lateral parts of the box $(-L, L)^d \times [0, T]$. For this, define

$$S(L,T) := \{(x,t) \in \mathbb{Z}^d \times [0,T] : |x|_\infty = L\},$$

the boundary of the box $(-L, L)^d \times [0, T]$ and define ${}_LH := \bigcup_{t \geqslant 0} {}_LH_t \times \{t\}$. For any $A \subset \mathbb{Z}^d$, let $N_S^A(L,T)$ be the cardinal of the set

$$\{(x,t) \in S(L,T) \cap {}_LH^A : (x_1,s_1),(x_2,s_2) \in S(L,T) \cap {}_LH^A \text{ such that } |s_1 - s_2| \geqslant 1\}.$$

Proposition 2.6.4. *Let $(L_j)_{j\geqslant 1}$ and $(T_j)_{j\geqslant 1}$ be two increasing sequences of integers. For any integers M, N, n,*

$$\lim_{j\to\infty} \mathbb{P}_r(N^{[-n,n]^d}(L_j,T_j) \leqslant M)\mathbb{P}_r(|_{L_j} H_{T_j}^{[-n,n]^d} \leqslant N) \\ \leqslant \mathbb{P}_r(H_t^{[-n,n]^d} = \varnothing \text{ for some } t) \quad (2.6.6)$$

Proof. Let $\mathcal{F}_{L,T}$ be the sigma-algebra generated by the Poisson processes of the graphical representation of the process $(\eta_t)_{t\geqslant 0}$ in $(-L, L)^d \times [0, T]$. For each site of ${}_LH_T^{[-n,n]^d}$, there is a probability at least

$$(1 + 2d\lambda_1)^{-1}$$

that a site does not give birth (exponential clock with parameter 1 associated to a death ringing before an exponential clock associated to a birth). By independance of the Poisson processes, the probability that none of $x \in {}_LH_T{}^{[-n,n]^d}$ contributes to the survival of the process is at least

$$\Big((1 + 2d\lambda_1)^{-1}\Big)^{|_LH_T^{[-n,n]^d}|}.$$

For the lateral parts of $(-L, L)^d \times [0, T]$, consider now a segment $\{x\} \times [0, T]$, where $|x|_\infty = L$, and define $(x, t_1), ..., (x, t_j)$ a maximal set of 1-sparse time-wise points of the segment in $S(L, T) \cap [-n, n]^d$ i.e. such that for any points (x, t_i) and (x, t_j) in this set, then $|t_i - t_j| \geqslant 1$. Fix $j \geqslant 1$, the segment

$$I = \bigcup_{k=1}^{j} \{x\} \times (t_k - 1, t_k + 1).$$

is of Lebesgue-measure at least $2j$. There is no arrow in the graphical representation starting from a site of I with probability at least

$$\Big(e^{-2j\lambda_1}\Big)^{2d}.$$

For each interval of length y in $\{x\} \times [0,T)\backslash I$, the event no arrow occurs or an arrow occuring is preceded by a death or a slowdown symbol, occurs with probability at least

$$\frac{1}{1+2d\lambda_1}.$$

Consequently, no points of $\{x\} \times [0,T]$ contributes to the survival of the process with probability at least

$$e^{-4dj\lambda_1}\Big(\frac{1}{2d\lambda_1}\Big)^j.$$

Counting the contribution of all such x,

$$\mathbb{P}_r(H_t^{[-n,n]^d} = \varnothing \text{ for some t}|\mathcal{F}_{L,T}) \geqslant e^{-4dk\lambda_1}\Big(\frac{1}{1+2d\lambda_1}\Big)^k \text{ a.s.} \tag{2.6.7}$$

on the event $\{N_S^{[-n,n]^d}(L,T) + |{}_LH_T^{[-n,n]^d}| \leqslant k\}$.

Then, consider two increasing sequences $(T_j)_{j\geqslant 0}$, $(L_j)_{j\geqslant 0}$ and integers M, N, define

$$H_j := \{N_S^{[-n,n]^d}(L_j,T_j) + |{}_LH_{T_j}^{[-n,n]^d}| \leqslant M+N\}.$$

If $G = \{H_t^{[-n,n]^d} = \varnothing \text{ for some } t\}$, by the martingale convergence theorem,

$$\lim_{j\to\infty} \mathbb{P}_r(G|\mathcal{F}_{L_j,T_j}) = \mathbf{1}_G \text{ a.s.}$$

From (2.6.7), for all $j \geqslant 0$, on H_j, $\mathbb{P}_r(G|\mathcal{F}_{L_j,T_j})$ is bounded below by some positive form and thus, $\overline{\lim_{j\to\infty}} H_j \subset G$. That is,

$$\overline{\lim_{j\to\infty}} \mathbb{P}_r(H_j) \leqslant \mathbb{P}_r(\overline{\lim_{j\to\infty}} H_j) \leqslant \mathbb{P}_r(G).$$

Furthermore, by FKG inequality (C),

$$\begin{aligned}
&\mathbb{P}_r(N^{[-n,n]^d}(L,T) + |{}_LH_T^{[-n,n]^d}| \leqslant M+N) \\
&\quad\geqslant \mathbb{P}_r(N_S^{[-n,n]^d}(L,T) \leqslant M, |{}_LH_T^{[-n,n]^d}| \leqslant N) \\
&\quad\geqslant \mathbb{P}_r(N_S^{[-n,n]^d}(L,T) \leqslant M)\mathbb{P}_r(|{}_LH_T^{[-n,n]^d}| \leqslant N)
\end{aligned}$$

this concludes the proof. □

As for the top of the box (Proposition 2.6.3), one can control the number of occupied sites on the lateral parts of the box $(-L,L)^d \times [0,T]$. For this, introduce for $i = 1, ..., d2^d$ and $u \in \{-,+\}^d$, the 2^d sides of the box by

$$S_i^u(L,T) := \{(x,t) \in \mathbb{Z}^d \times [0,T], x_i = u_iL, sgn(x_j) = u_j \ (j \neq i)\}$$

and $N_i^A(L,T)$ the cardinal of the set

$$\{(x,t) \in S_i^u(L,T) \cap {}_LH^A : (x_1,s_1),(x_2,s_2) \in S(L,T) \cap {}_LH^A \text{ such that } |s_1-s_2| \geqslant 1\}.$$

By symmetry and reflexion with respect to the time axis, it is enough to look only at the positive coordinates :

$$S_+(L,T) := S_1^{(+,\dots,+)}(L,T) = \{(x,t) \in \mathbb{Z}^d \times [0,T], x_1 = L, x_j \geqslant 0 \ (j \neq i)\}.$$

Proposition 2.6.5. *For any integers M, L, T and $n < L$,*

$$\mathbb{P}_r(N_+^{[-n,n]^d}(L,T) \leqslant M)^{d2^d} \leqslant \mathbb{P}_r(N_S^{[-n,n]^d}(L,T) \leqslant Md2^d)$$

Proof. Remark that $\{N_i^{[-n,n]^d}(L,T), 1 \leqslant i \leqslant d2^d\}$ are identically distributed and positively correlated. Moreover,

$$N_S^{[-n,n]^d}(L,T) \leqslant \sum_{i=1}^{d2^d} N_i^{[-n,n]^d}(L,T).$$

So, as for Proposition 2.6.4, one has by FKG inequality,

$$\begin{aligned}
\mathbb{P}_r(N_+^{[-n,n]^d}(L,T) \leqslant M)^{d2^d} &= \prod_{i=1}^{d2^d} \mathbb{P}_r(N_i^{[-n,n]^d}(L,T) \leqslant M) \\
&\leqslant \mathbb{P}_r(\bigcap_{i=1}^{d2^d} N_i^{[-n,n]^d}(L,T) \leqslant M) \\
&\leqslant \mathbb{P}_r(N_S^{[-n,n]^d}(L,T) \leqslant Md2^d).
\end{aligned}$$

□

Space-time conditions

Proposition 2.6.6. *Suppose $(\eta_t)_{t\geqslant 0}$ survives. For any $\epsilon_6 > 0$, there exist integers $n, L, T > 0$ such that*

$$\mathbb{P}_r\left({}_{L+2n}H_{T+1}^{[-n,n]^d} \supset x + [-n,n]^d \textit{ for some } x \in [0,L)^d \right) > 1 - \epsilon_6 \tag{2.6.8}$$

and

$$\mathbb{P}_r\left({}_{L+2n}H_{t+1}^{[-n,n]^d} \supset x + [-n,n]^d \textit{ for some } (x,t) \in \{L+n\} \times [0,L)^{d-1} \times [0,T) \right) > 1 - \epsilon_6 \tag{2.6.9}$$

Proof. Fix $\delta > 0$. By Proposition 2.6.1, choose n such that

$$\mathbb{P}_r(H_t^{[-n,n]^d} \neq \varnothing \ \forall t \geqslant 0) > 1 - \delta^2$$

Let N be sufficiently large so that N points in $\mathbb{Z}^d$ contain at least N' points which are $(2n+1)$-sparse in L_∞-distance. Choose now N' sufficiently large so that

$$\Big[1 - \mathbb{P}_r({}_{n+1}H_t^{\{0\}} \supset [-n,n]^d)\Big]^{N'} \leqslant \delta.$$

Likewise, choose M sufficiently large so that M points in $\mathbb{Z}^d$ contain at least M' points which are $(2n+1)$-sparse. Choose now M' sufficiently large so that

$$\Big[1 - \mathbb{P}_r\big({}_{n+1}H_1^{\{0\}} \supset [0,2n] \times [-n,n]^{d-1}\big)\Big]^{M'} \leqslant \delta.$$

Fix n, L, N, the map $t \mapsto \mathbb{P}_r(|{}_LH_t^{[-n,n]^d}| \geqslant 2^d N)$ is continuous and $\lim\limits_{n\to\infty} \mathbb{P}_r(|{}_LH_t^{[-n,n]^d}| > 2^d N) = 0$, by Proposition 2.6.2, there exist two increasing sequences $L_j \uparrow \infty$ and $T_j \uparrow \infty$ such that for all $j \geqslant 1$,

$$\mathbb{P}_r(|{}_{L_j}H_{T_j}^{[-n,n]^d}| > 2^d N) = 1 - \delta.$$

Using Proposition 2.6.4, there exists some j_0 for which,

$$\mathbb{P}_r(N_S^{[-n,n]^d}(L_{j_0}, T_{j_0}) > Md2^d) > 1 - \delta.$$

Considering $L = L_{j_0}$ and $T = T_{j_0}$, applying Propositions 2.6.3 and 2.6.5, one has

$$\mathbb{P}_r(|{}_LH_T^{[-n,n]^d} \cap [0,L)^d| > 2^d N) \geqslant 1 - \delta^{1/2^d}$$

and

$$\mathbb{P}_r(N_S^{[-n,n]^d}(L,T) > Md2^d) > 1 - \delta^{1/d2^d}.$$

In other words, because the Poisson processes used in the graphical representation are independent in different space-time regions,

$$\mathbb{P}_r\left({}_{L+2n}H_{T+1}^{[-n,n]^d} \supset x + [-n,n]^d \text{ for some } x \in [0,L)^d\right) \geqslant (1 - \delta^{1/d2^d})(1-\delta)$$

and

$$\mathbb{P}_r\Big({}_{L+2n}H_{T+1}^{[-n,n]^d} \supset x + [-n,n]^d \text{ for some } (x,t) \in \{L+n\} \times [0,L)^{d-1} \times [0,T)\Big) \geqslant (1 - \delta^{1/2^d})(1-\delta).$$

Conclude by choosing δ such that $(1-\delta^{1/2d})(1-\delta) \geqslant 1-\epsilon$ and $(1-\delta^{1/d2^d})(1-\delta) \geqslant 1-\epsilon$. $\square$

Proposition 2.6.7. *Suppose* (2.6.8)-(2.6.9) *are satisfied. Then, for any* $\epsilon_7 = \epsilon_7(\epsilon_6) > 0$, *there exist* n, L, T *such that*

$$\mathbb{P}_r\big({}_{2L+3n}H_t^{[-n,n]^d} \supset x + [-n,n]^d \text{ for some } (x,t) \in [L+n, 2L+n] \times [0,2L)^{d-1} \times [T, 2T]\big) > 1 - \epsilon_7. \quad (2.6.10)$$

Proof. For any $\epsilon_7 > 0$, choose n, L and T as in (2.6.8)-(2.6.9), by Proposition (2.6.6). With (2.6.9), with $\mathbb{P}_r-$probability at least $1-\epsilon_6$, there exists $(x,t) \in \{L+n\} \times [0,L)^{d-1} \times [0,T)$ such that ${}_{L+2n}H_{t+1}^{[-n,n]^d} \supset x + [-n,n]^d$.

By the Markov property and (2.6.8), starting from $T+1$, with $\mathbb{P}_r-$probability at least $1-\epsilon_6$, there exists some y such that $y-x \in [0,L)^d$ satisfying ${}_{L+2n}H_{T+1}^{[-n,n]^d} \supset y+[-n,n]^d$. Consequently,

$$\mathbb{P}_r\big({}_{2L+3n}H_t^{[-n,n]^d} \supset x + [-n,n]^d \text{ for some } (x,t) \in [L+n, 2L+n] \times [0,2L)^{d-1} \times [T+1, 2(T+1)]\big) \geqslant (1-\epsilon_6)^d.$$

□

The next result links the previous estimates with a percolation process.

Block constructions The following two constructions rely on the geometry of the boxes only, proofs are similar to the ones of [5, Lemma 18] and [5, Lemma 19] respectively. They allow us to position the successive boxes adjacently and well centred.

Proposition 2.6.8. *Suppose* $(\eta_t)_{t\geqslant 0}$ *survives. For any* $\epsilon_8 = \epsilon_8(\epsilon_7) > 0$ *and fix* $k \in \mathbb{N}$, *there exist integers* n, a, b *such that* $n < a$ *for which : for all* $(x,s) \in [-a,a]^d \times [0,b]$, *with* $\mathbb{P}_r-$*probability at least* $1-\epsilon_8$, *there exists a translate* $(y,t) + [-n,n]^d \times \{0\}$ *satisfying :*

i. $(y,t) \in [a,3a] \times [-a,a]^{d-1} \times [5b,6b]$.

ii. From $(x,s) + [-n,n]^d \times \{0\}$, *there exist active paths reaching any points of* $(y,t) + [-n,n]^d \times \{0\}$ *lying within the region*

$$[-5a, 5a]^d \times [0, 6b].$$

The idea is to repeat sufficiently enough the Proposition 2.6.7 in order to translate the center (x,s) of a box to the center (y,t) of another box, so that if the first box is occupied, then the second one is as well and so on [see Figure 2.3].

Proof. Choose n, L, T as in Proposition 2.6.7 . Define $a = 2L+n$ and $b = 2T$. One can thus construct boxes as following : noting one needs to recentre within the box $(y,t) \in [a,3a] \times [-a,a]^{d-1} \times [5b,6b]$:

(1) for $2 \leqslant i \leqslant d$, for some current centre (z,r) such that $z_i \geqslant 0$ or $z_i < 0$, it suffices to move it in the opposite direction. Since $a \geqslant 2L$, the ith coordinate does not leave out of $[-a,a]$.

(2) Move the spatial coordinate to reach $[a, 3a]$. Since it always moves by at least $2L + n$ and $2L + n \geqslant 2a$, it reaches $[a, 3a]$ in at most four steps.

(3) Move the time coordinate to reach $5b$. As it moves between T and $2T$, it reaches $5b$ after four to ten steps. As $b = 2T$, it does not overcross $6b$ by 10 steps.

As each step depend only of Poisson processes within the region $[-5a, 5a]^d$ by disjoint time intervals, by Proposition 2.6.7, this construction succeeds with probability at least $(1 - \epsilon_7)^{10} =: 1 - \epsilon_8$. □

Iterating k times the previous result, one obtains (see Figure 2.3) :

Proposition 2.6.9. *Suppose $(\eta_t)_{t\geqslant 0}$ survives. For any $\epsilon_9 = \epsilon_9(\epsilon_8) > 0$ and $k \in \mathbb{N}$ fixed, there exist $\delta > 0$, and integers n, a, b such that $n < a$ for which : For all $(x, s) \in [-a, a]^d \times [0, b]$, with $\mathbb{P}_r$−probability at least $(1-\epsilon_9)^k$, there exists a translated $(y, t) + [-n, n]^d \times \{0\}$ such that :*

i. $(y, t) \in (2ka + [-a, a]) \times [-a, a]^{d-1} \times (5kb + [0, b])$.

ii. From $(x, s) + [-n, n]^d \times \{0\}$, there exist active paths reaching any point of $(y, t) + [-n, n]^d \times \{0\}$ lying within the region

$$\mathcal{R} = \bigcup_{j=0}^{k-1} (2ja + [-5a, 5a]) \times [-5a, 5a]^{d-1} \times (5jb + [0, 6b]).$$

Comparison with oriented percolation For $p, q \in \mathbb{Z}$ such that $q \geqslant 0$ and $p + q$ is even, define

$$v_{p,q} := [-a, a]^d \times [0, b] + (p2kae_d \times q5kb)$$

and

$$\mathcal{S} = \bigcup_{q\geqslant 0} \bigcup_{p+q \ \text{even}} \Big(\mathcal{T} + (p2kae_d \times q5kb)\Big),$$

where $\mathcal{T} = [-a, a]^{d-1} \times \Big\{(x_d, t) \in \mathbb{Z} \times \mathbb{R}_+ : 0 \leqslant t \leqslant (5k + 1)b,\ -5a \pm at/b \leqslant x_d \leqslant 5a \pm at/b\Big\}$. Here, $\mathcal{S}$ is a cross shaped nesting of successive boxes (as in Figure 2.3) using reflections and symmetries. Similarly to [5, Lemma 21], one has

Theorem 2.6.1. *If $(\eta_t)_{t\geqslant 0}$ survives, there exist integers n, a such that*

$$\mathbb{P}_r(\eta_t^{[-n,n]^d} \ \text{survives in} \ \mathbb{Z} \times [-5a, 5a]^{d-1} \times [0, \infty)) > 0$$

Proof. Adapting the proof of [5, Lemma 21], fix $\delta > 0$ and $\epsilon > 0$ such that $1 - \epsilon > 1 - \delta$. Choose n, a, b as in Proposition 2.6.9. Construct random variables $\{Z_n(i) = (I_n(i), P_n(i)) : n \geqslant 0, i \geqslant 0\}$, where $I_n(i) \in \{0, 1\}$ and $P_n(i) \in \mathbb{Z}^d \times [0, \infty)$ such that $P_n(i)$ is undefined if $I_n(i) = 0$. Fix $Z_0(0) = (1, 0)$.

For defined random variables $\{Z_n(i) : n \leqslant N, i \geqslant 0\}$, construct recursively $Z_{N+1}(i) = (I_{N+1}(i), P_{N+1}(i))$ as follows.

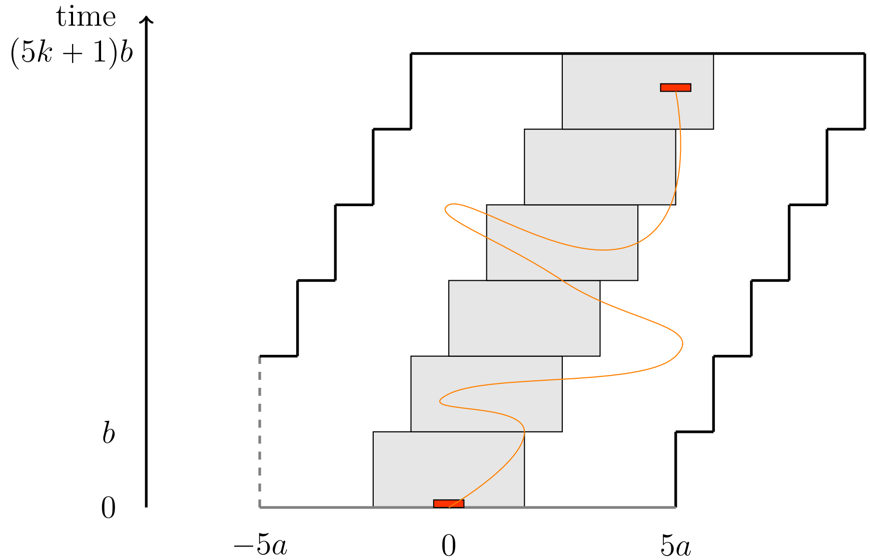

FIGURE 2.3: Set $d = 1$. The targeted region goes upward-rightward (reflections are not drawn, but a symmetric draw going upward-leftward does exist as the dashed line suggests it). Occupied translated sets $[-n, n]^d$ stand in the gray regions and are reached by paths lying in the area delimited by the stair shaped plain lines.

1. the random variable $I_{N+1}(i)$ is 1 if : for $j \in \{i, i-1\}$, $I_N(j) = 1$, $P_N(j) + [-n, n]^d$ is joined to every point of some translate of $[-n, n]^d$ centred in $v_{2i-N-1,N+1}$ within $\mathcal{S}$.
2. if $I_{N+1}(i) = 1$ then $P_{N+1}(i)$ is defined as the centre of some translate of $[-n, n]^d$.

With this construction, if for any n, $\{i \geqslant 0 : I_n(i) = 1\} \neq \varnothing$ then the process $(\eta_t^{[-n,n]^d})_{t \geqslant 0}$ survives in $\mathbb{Z} \times [-5a, 5a]^{d-1} \times [0, \infty)$. It remains then to show

$$\mathbb{P}_r(\{i \geqslant 0 : I_n(i) = 1\} \neq \varnothing \ \forall n \geqslant 0) > 0.$$

Define $\mathcal{F}_n = \sigma(Z_k(i), i \geqslant 0, 0 \leqslant k \leqslant n)$ and by Proposition 2.6.9 :

$$\mathbb{P}_r(I_{n+1}(i) = 1 | \mathcal{F}_n) > 1 - \delta \text{ on } \{I_n(i-1) = 1 \text{ or } X_n(i) = 1\}$$

But conditionally to $\mathcal{F}_n$, variables $\{I_n(i), i \geqslant 0\}$ are 1-dependent. By Theorem 2.2.6, one can construct Bernoulli random variables whose product measure of density p is lower than the distribution of the 1-dependent variables. By taking $1-p$ close to 1, by Lemma 2.2.5, one has $I_n(i) = 1$ for an infinity of pairs (n, i) with positive probability. □

2.6.2 Extinction of the critical case

Using the foregoing dynamic block construction, one concludes to the Theorem 2.2.3 :

Proof of Theorem 2.2.3. Let $r > 0$ be such that the process $(\eta_t)_{t\geqslant 0}$ survives. In the block constructions established in Propositions 2.6.8 and 2.6.9, each event depends only on the graphical representation of the process in each box $(2ja+[-5a,5a])\times[-5a,5a]^{d-1}\times(5jb+[0,6b])$, for $j \geqslant 0$. Then, Propositions 2.6.8 and 2.6.9 are preserved with $\mathbb{P}_{r+\delta}$-probability for some $\delta > 0$. From Theorem 2.6.1, the process $(\eta)_{t\geqslant 0}$ survives in $r+\delta$. But since $r+\delta \leqslant r_c$, then $r < r_c$. That is, whenever the process survives, r stands below r_c : the critical process dies out. □

2.7 The mean-field model

Consider in this section the mean-field model associated to the multitype process, both symmetric and asymmetric. This is a deterministic and non-spatial process where all individuals are mixed up, leading to study the densities of each type of particles overall.

Mean-field models give rise to differential systems and are interesting to compare stochastic behaviours, as previously studied, with corresponding deterministic behaviours. We investigate here the equilibria of these differential systems, first in the asymmetric model, and in the symmetric model then. Since existence of such equilibria yields the existence of a critical value, we survey the mean-field equations in order to exhibit conditions on the parameter r to deduce bounds on the critical value r_c.

Subsequently, let u_i be the density of type-i individuals for $i = 1,2,3$. Overall, one has $u_1+u_2+u_3 = 1-u_0$. Furthermore, in connection with the definition of wild and sterile individuals, we consider as well v_1, resp. v_2, the density of the wild individuals (type-1 and type-3 individuals), resp. the sterile individuals (type-2 and type-3 individuals), and the density of empty sites $v_0 = u_0$. Relations between the u-system and the v-system are described by

$$\begin{cases} u_1 = 1 - v_0 - v_2 \\ u_2 = 1 - v_0 - v_1 \\ u_3 = v_0 + v_1 + v_2 - 1 \end{cases} . \tag{2.7.1}$$

Since we consider densities, both systems satisfy

$$u_i \in [0,1] \text{ for } i = 0,1,2,3, \qquad v_i \in [0,1] \text{ for } i = 0,1,2. \tag{2.7.2}$$

2.7.1 Asymmetric multitype process

Assuming total mixing, the mean-field model associated to the asymmetric multitype process is given by :

$$\begin{cases} u_1' = 2d(\lambda_1 u_1 + \lambda_2 u_3)u_0 + u_3 - (r+1)u_1 \\ u_2' = ru_0 + u_3 - u_2 \\ u_3' = ru_1 - 2u_3 \end{cases} . \tag{2.7.3}$$

This system admits two equilibria :

$$(u_1, u_2, u_3) = \Big(0, \frac{r}{r+1}, 0\Big),$$

$$\Big(\frac{1}{r+1} - \frac{r+2}{4d\lambda_1 + 2d\lambda_2 r}, \frac{r}{r+1} - \frac{r}{2}\Big(\frac{1}{r+1} - \frac{r+2}{4d\lambda_1 + 2d\lambda_2 r}\Big), \frac{r}{2(r+1)} - \frac{r(r+2)}{2(4d\lambda_1 + 2d\lambda_2 r)}\Big).$$

Note that the first equilibrium gives $(u_1, u_2, u_3) = \Big(0, \frac{r}{r+1}, 0\Big)$ which puts a positive density on the sterile individuals and none on the others, which corresponds to the extinction of the process.

$$\begin{cases} v_0' = -2d\Big((\lambda_2 - \lambda_1)v_0 + \lambda_2 v_1 + (\lambda_2 - \lambda_1)v_1 + \lambda_1 - \lambda_2\Big)v_0 - (r+2)v_0 - v_1 - v_2 + 2 \\ v_1' = 2d\Big((\lambda_2 - \lambda_1)v_0 + \lambda_2 v_1 + (\lambda_2 - \lambda_1)v_2 + \lambda_1 - \lambda_2\Big)v_0 - v_1 \\ v_2' = r(1 - v_2) - v_2 \end{cases} \tag{2.7.4}$$

This system gives rise to an equilibrium :

$$(v_0, v_1, v_2) = \Big(\frac{2+r}{4d\lambda_1 r + 2d\lambda_2 r}, \frac{r+2}{2(r+1)} - \frac{(r+2)^2}{2(4d\lambda_1 + 2d\lambda_2 r)}, \frac{r}{r+1}\Big).$$

In particular, by checking conditions (2.7.2), one highlights a condition : the density v_1 is non-negative as soon as

$$4d\lambda_1 + 2d\lambda_2 r > (r+1)(r+2).$$

which gives the following condition

$$r < \frac{2d\lambda_2 - 3 + \sqrt{(2d\lambda_2 - 3)^2 - 8(1 - 2d\lambda_1)}}{2} \tag{2.7.5}$$

This indicates a lower bound for the phase transition.

2.7.2 Symmetric multitype process

The mean-field equations associated to the symmetric multitype process are :

$$\begin{cases} u_1' = 2d(\lambda_1 u_1 + \lambda_2 u_3)u_0 + u_3 - (r+1)u_1 \\ u_2' = ru_0 + u_3 - u_2 - 2d(\lambda_1 u_1 + \lambda_2 u_3)u_2 \\ u_3' = ru_1 + 2d(\lambda_1 u_1 + \lambda_2 u_3)u_2 - 2u_3 \end{cases} \tag{2.7.6}$$

As previously, this system admits one trivial equilibrium :

$$(u_1, u_2, u_3) = \Big(0, \frac{r}{r+1}, 0\Big)$$

retrieving once again a situation related to the extinction of the process, by a positive density of sterile individuals and none of the wild ones. We derive the non-trivial equilibrium thanks to the corresponding v-system :

$$\begin{cases} v_0' = -2d\Big((\lambda_2-\lambda_1)v_0 + \lambda_2 v_1 + (\lambda_2-\lambda_1)v_1 + \lambda_1 - \lambda_2\Big)v_0 - (r+2)v_0 - v_1 v_2 + 2 \\ v_1' = 2d\Big((\lambda_2-\lambda_1)v_0 + \lambda_2 v_1 + (\lambda_2-\lambda_1)v_2 + \lambda_1 - \lambda_2\Big))(1-v_1) - v_1 \\ v_2' = r(1-v_2) - v_2 \end{cases} \tag{2.7.7}$$

Let us determine the non-trivial equilibrium. Last line of (2.7.7) gives already $v_2 = \dfrac{r}{r+1}$. Using relations of (2.7.1) in (2.7.7), according to $v_1' = 0$, an equilibrium (v_0, v_1, v_2) satisfies in particular

$$v_1 = 2d(\lambda_1 u_1 + \lambda_2 u_3)(1-v_1) \tag{2.7.8}$$

checking v_1 cannot be equal to 1, one then has

$$\frac{v_1}{1-v_1} = 2d(\lambda_1 u_1 + \lambda_2 u_3) \tag{2.7.9}$$

and

$$v_1 \neq 1. \tag{2.7.10}$$

On the other hand, from the u-system (2.7.6) with relations (2.7.1) and using condition (2.7.10),

$$\begin{aligned} u_1' = 0 &\Leftrightarrow \frac{v_1 v_0}{1-v_1} + (2+r)v_0 + v_1 - \frac{r+2}{r+1} = 0 \\ u_2' = 0 &\Leftrightarrow (r+2)v_0 + v_1 - \frac{r+2}{r+1} + \frac{v_0 v_1}{1-v_1} \\ u_3' = 0 &\Leftrightarrow (1-v_1)^2 + (1-v_1)(\frac{1}{r+1} - (r+1)v_0) - v_0 = 0 \end{aligned}$$

By solving the last line with respect to $(1-v_1)$, one has

$$1 - v_1 = (r+1)v_0 \text{ or } 1 - v_1 = -\frac{1}{r+1}.$$

To deduce the value of v_1, we investigate both possibilities. Using (2.7.9)

1. if $1 - v_1 = -\dfrac{1}{r+1}$, But since this value is negative, necessarily $1 - v_1 \neq -\dfrac{1}{r+1}$.
2. if $1 - v_1 = (r+1)v_0$, using (2.7.9) v_0 solves

$$2d(\lambda_1 + \lambda_2 r)(r+1)v_0^2 - (2d\lambda_1 + 2d\lambda_2 r + r + 1)v_0 + 1 = 0.$$

This implies

$$v_0 = \frac{1}{r+1} \text{ or } v_0 = \frac{1}{2d\lambda_1 + 2d\lambda_2 r}.$$

(a) if $v_0 = \frac{1}{r+1}$, then by relations (2.7.1),

$$u_1 = 0,\ u_3 = 1,\ v_2 = 1 + u_2,$$

which is a contradiction.

(b) if $v_0 = \dfrac{1}{2d\lambda_1 + 2d\lambda_2 r}$, then

$$v_1 = \frac{r+1}{2d(\lambda_1 + \lambda_2 r)},\ v_2 = \frac{r}{r+1}.$$

Verifying this v-system to be a set of densities by condition 2.7.2, one case highlights a condition on r : $v_1 \leqslant 1$ if and only if $r(1 - 2d\lambda_2) \leqslant 2d\lambda_1 - 1$. In the case where $\lambda_2 \leqslant 1/(2d)$, then one has the condition

$$r \leqslant \frac{2d\lambda_1 - 1}{1 - 2d\lambda_2}. \tag{2.7.11}$$

Consequently, a non-trivial equilibrium of 2.7.7 is given by

$$(v_0, v_1, v_2) = \Big(\frac{1}{2d\lambda_1 + 2d\lambda_2 r}, \frac{r+1}{2d\lambda_1 + 2d\lambda_2 r}, \frac{r}{r+1}\Big) \tag{2.7.12}$$

To put in a nutshell, this survey of equilibria associated to both mean-field models, in symmetric and asymmetric case, gave us the bounds (2.7.5) and (2.7.11) for the phase transition.

We will turn into a rigorous proof of the convergence of the empirical densities to these reaction-diffusion systems. This is dealt with the hydrodynamic limits in Chapters 4 and 5.

3

Survival and extinction conditions in quenched environment

Contents

3.1 Introduction

One considers here the unidimensional contact process on $\mathbb{Z}$, either in an inhomogeneous (deterministic) environment, or in a quenched random environment where the growth parameter takes two possible values depending on the environment. Previously in Chapter 2, we investigated the contact process in a dynamic random environment, for which we exhibited a phase transition. Nevertheless, through the use of percolation theory, we were not able to explicit rigorous numerical bounds on the phase transition, but we are now.

Here, we are concerned by two kinds of quenched random environment on $\mathbb{Z}$: in the first case, growth rates are randomly chosen according to each vertex ; in the second case, growth rates are chosen randomly on each oriented edges.

The contact process in random environment has already been studied in many ways to understand how a random rate affects the behaviour of the process. In an unidimensional case, M. Bramson, R. Durrett and R. Schonmann [12] exhibited an intermediary phase where the process survives without growing linearly. In higher dimensions, N. Madras, R. Schinazi and R. Schonmann [60] showed there exist choices of a random death rate for which the critical contact process survives. Several survival and extinction conditions have been given successively by T.M. Liggett [52, 53], C. Newman and S. Volchan [66] in dimension 1 and E. Andjel [1], A. Klein [45] in higher dimensions.

We will rely on [52, 53] whose model and results are described in Section 3.2 before taking advantage of them by illustrating them in our framework. We expose our results when growth rates are depending on vertices in Section 3.3 and depending on edges in Section 3.4. To conclude the chapter, we obtain by the two previous sections a list of numerical bounds in Section 3.5

3.2 Settings and results

3.2.1 Preliminaries

The *contact process in random environment* introduced by T.M. Liggett [52, 53] is a Markov process $(\chi_t)_{t\geqslant 0}$ on $\{0,1\}^{\mathbb{Z}}$ whose transitions at each site $x \in \mathbb{Z}$ are given by

$$\begin{aligned} &0 \to 1 \text{ at rate } \rho(x)\chi(x+1) + \lambda(x)\chi(x-1) \\ &1 \to 0 \text{ at rate } \delta(x) \end{aligned} \tag{3.2.1}$$

where the family $\{\big(\delta(x), \rho(x), \lambda(x)\big), x \in \mathbb{Z}\}$ stands for the random environment which is an ergodic stationary process. See Figure 3.1. If $\{\big(\delta(x), \rho(x), \lambda(x)\big), x \in \mathbb{Z}\}$ is chosen deterministic, hence inhomogeneous, we will refer to it as the *inhomogeneous contact process*.

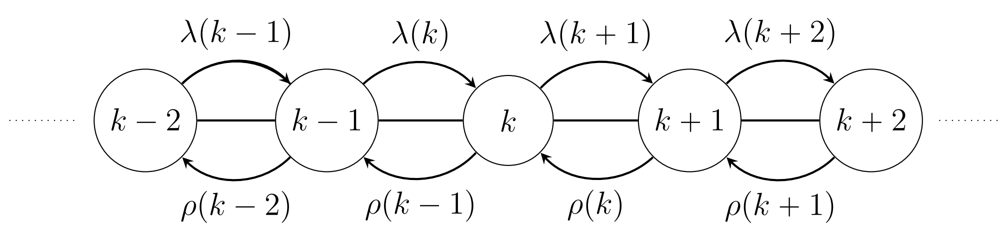

Figure 3.1: Quenched random environment

Definition 3.2.1. Let ω be the random environment. For almost-every realization of ω, the process $(\chi_t)_{t\geqslant 0}$ *survives* if

$$\mathbb{P}^\omega(\forall t \geqslant 0,\ X_t \neq \varnothing) > 0$$

and *dies out* if

$$\mathbb{P}^\omega(\forall t \geqslant 0,\ X_t \neq \varnothing) = 0.$$

T.M. Liggett [52, 53] settled survival and extinction conditions for such a process, among which :

Theorem 3.2.1 (T.M. Liggett [52])**.** *1. The inhomogeneous contact process dies out if for all $n \geqslant 0$,*

$$\sum_{k\geqslant n}\prod_{j=n}^{k}\frac{\rho(j)}{\delta(j+1)} < \infty \quad \textit{and} \quad \sum_{k\leqslant n}\prod_{j=k}^{n}\frac{\lambda(j)}{\delta(j-1)} < \infty.$$

2. The contact process in random environment dies out if

$$\mathbb{E}\log\rho(0) < \mathbb{E}\log\delta(0) \ \textit{and}\ \mathbb{E}\log\lambda(0) < \mathbb{E}\log\delta(0).$$

3. Suppose the random vector $\{(\delta(k), \rho(k), \lambda(k)), k \in \mathbb{Z}\}$ i.i.d. The contact process in random environment dies out if

$$\mathbb{E}\Big(\rho(k)\delta(k)^{-1}\Big) < 1$$

and

$$\mathbb{E}\delta(k)^{-1}\Big(1-\mathbb{E}\Big(\delta(k)\lambda(k)^{-1}\Big)\Big) < \mathbb{E}\lambda(k)^{-1}\Big(1-\mathbb{E}\Big(\rho(k)\delta(k)^{-1}\Big)\Big).$$

Theorem 3.2.2 (T.M. Liggett [53])**.** *The contact process in random environment survives if the two following series converge,*

$$\sum_{j\geqslant 0}\mathbb{E}\left(\frac{1}{\lambda(j+1)}\prod_{k=1}^{j}\frac{\delta(k)\Big(\lambda(k)+\rho(k-1)+\delta(k)\Big)}{\lambda(k)\rho(k-1)}\right),$$

$$\sum_{j\geqslant 0}\mathbb{E}\left(\frac{1}{\rho(j-1)}\prod_{k=1}^{j}\frac{\delta(k-1)\Big(\lambda(k)+\rho(k-1)+\delta(k-1)\Big)}{\lambda(k)\rho(k-1)}\right).$$

Furthermore, if $\{(\delta(k), \rho(k), \lambda(k)), k \in \mathbb{Z}\}$ is i.i.d., then the contact process in random environment survives if

$$\mathbb{E}\frac{\delta(\lambda+\rho+\delta)}{\lambda\rho} < 1.$$

Adaptating the results above of T.M. Liggett [52, 53], one is able to exhibit extinction and survival conditions leading us to explicit numerical bounds on the phase transition of the contact process in quenched random environment.

3.2.2 The model

Our framework is the following. One describes the environment as a configuration over the sites of $\mathbb{Z}$. Let $p \in (0,1)$, define a random environment $\omega \in \{0,1\}^{\mathbb{Z}}$ where each site $x \in \mathbb{Z}$ is free (0) with probability $1-p$ or slowed-down (1) with probability p, independently of any other site.

The contact process in random environment we consider here is a contact process $(\chi_t)_{t\geqslant 0}$ with state space $\{0,1\}^{\mathbb{Z}}$ and quenched environment ω. Let λ_1 and λ_2 be growth parameters such that

$$\lambda_2 \leqslant \lambda_c < \lambda_1, \tag{3.2.2}$$

where λ_c stands for the critical growth rate of the basic contact process on $\{0,1\}^{\mathbb{Z}}$. Recall from previous chapter that some release rate r was curbing the expansion of a supercritical contact process with $\lambda_1 > \lambda_c$ to a subcritical rate $\lambda_2 \leqslant \lambda_c$. Subsequently, for $r \in (0,\infty)$,

$$p = r/(r+1) \tag{3.2.3}$$

stands for (in connection with the previous chapter) the probability a site is slowed down (corresponding to the minimum of two exponential clocks with parameters r and 1). Deaths occur at rate 1.

The process $(\chi_t)_{t\geqslant 0}$ is still monotone according to Chapter 2 Section 2.4.

Denote by $\mathbb{P}^{\omega}_{\lambda_1,\lambda_2,r}$ the distribution of $(\chi_t)_{t\geqslant 0}$ with parameters (λ_1,λ_2,r) and environment ω. For fixed parameters λ_1 and λ_2 satisfying $\lambda_2 \leqslant \lambda_c < \lambda_1$, simplify by $\mathbb{P}^{\omega}_{r}$. For any $A \subset \mathbb{Z}$, define $X_t^A := \{x \in \mathbb{Z} : \chi_t^A(x) = 1\}$, where χ_t^A denotes the process at time t started from the initial configuration $\chi_0 = \mathbf{1}_A$. If $A = \{0\}$, simplify by $X_t \equiv X_t^{\{0\}}$.

Consider subsequently two kinds of random environment : one depending of the vertices and one depending on the edges of the graph.

3.3 Random growth on vertices

Consider the dynamics where growth rates are affected to vertices. If $\lambda_v(k)$ is the growth rate from site $k \in \mathbb{Z}$: a birth at site k occurs at rate $\lambda_v(k-1)$ if $k-1$ is occupied plus at rate $\lambda_v(k+1)$ if $k+1$ is occupied, where

$$\lambda_v(k) = \lambda_1(1-\omega(k)) + \lambda_2\omega(k) \tag{3.3.1}$$

See Figure 3.2.

Based on the notations of Section 3.2, one has

$$\lambda(k+1) = \rho(k-1) = \lambda_v(k)$$

and $\lambda_v(k) = \rho_v(k)$ for all $k \in \mathbb{Z}$.

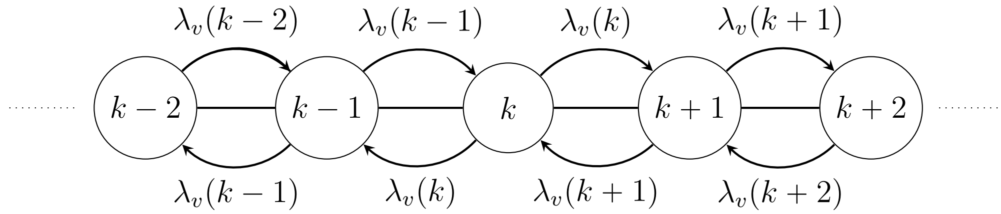

FIGURE 3.2: Random environment on vertices

3.3.1 Extinction conditions

Theorem 3.2.1-(1) can be rewritten as follows.

Theorem 3.3.1. *Assume that ω is a fixed environment. The inhomogeneous contact process $(\chi_t)_{t\geqslant 0}$ dies out if for all $n \in \mathbb{Z}$,*

$$\sum_{k\geqslant n}\prod_{j=n}^{k}\lambda_v(j+1) < \infty \text{ and } \sum_{k\leqslant n}\prod_{j=k}^{n}\lambda_v(j-1) < \infty. \tag{3.3.2}$$

where for $j \in \mathbb{Z}$, $\lambda_v(j)$ is defined by (3.3.1).

Proof. Introduce a modified version $(\alpha_t)_{t\geqslant 0}$ of the process $(\chi_t)_{t\geqslant 0}$ where a death at site $x \in \mathbb{Z}$ occurs uniquely if $\alpha(x-1) = 0$ or $\alpha(x+1) = 0$, while births occur at the same rate than $(\chi_t)_{t\geqslant 0}$:

$$0 \to 1 \text{ at rate } \sum_{y:|y-x|=1}\Big(\lambda_1(1-\omega(y)) + \lambda_2\omega(y)\Big)\alpha(y) \tag{3.3.3}$$

$$1 \to 0 \text{ at rate } \mathbf{1}\{n_0(x,\alpha) > 0\} \tag{3.3.4}$$

where $n_0(x,\alpha) = \sum_{y:|y-x|=1}\mathbf{1}\{\alpha(y) = 0\}$ stands for the number of neighbours of site x that are in state 0. This way, if initially the set $\{x \in \mathbb{Z} : \alpha_0(x) = 0\}$ is a non-empty interval then for all $t > 0$, $\{x \in \mathbb{Z} : \alpha_t(x) = 0\}$ is still an interval of $\mathbb{Z}$ until it potentially disappears in case α_t is identically equal to 1 on $\mathbb{Z}$. In the non-empty case, considering times at which a flip occurs, each end of this interval moves respectively as a birth and death chain : the rightmost zero evolves according to

$$k \to k+1 \text{ at rate } 1 \quad \text{and} \quad k \to k-1 \text{ at rate } \lambda_v(k+1)$$

and the leftmost zero evolves according to

$$k \to k+1 \text{ at rate } \lambda_v(k-1) \quad \text{and} \quad k \to k-1 \text{ at rate } 1$$

For $m, \ell \in \mathbb{Z}$ such that $m < 0 < \ell$, consider the initial condition

$$\alpha_0(x) = \begin{cases} 0 \text{ if } m \leqslant x \leqslant \ell, \\ 1 \text{ otherwise.} \end{cases}$$

Since both rightmost and leftmost zeros move as birth-death chains $\mathbb{Z}$, it remains to study their hitting time of 0. Define $(R_n)_{n\geqslant 0}$ and $(L_n)_{n\geqslant 0}$ two corresponding Markov chains, whose respective transition matrices Q^R and Q^L are defined by

$$Q^R(k,k+1) = \frac{1}{1+\lambda_v(k+1)},\ Q^R(k,k-1) = \frac{\lambda_v(k+1)}{1+\lambda_v(k+1)},$$
$$Q^L(k,k+1) = \frac{\lambda_v(k-1)}{1+\lambda_v(k-1)},\ Q^L(k,k-1) = \frac{1}{1+\lambda_v(k-1)}.$$

For $a \in \mathbb{Z}$, note P_a^R and P_a^L their respective probability measures conditionally in $R_0 = a$ and $L_0 = a$. Denote by $(S_n)_{n\geqslant 1}$ the flipping times and consider $(\alpha_n)_{n\geqslant 1}$, the skeleton-Markov chain corresponding to $(\alpha_t)_{t\geqslant 0}$, such that $\alpha_n = \alpha_{S_n}$ for all $n \geqslant 1$. Then,

$$\mathbb{P}_r^\omega(\alpha_n(x) = 0) = P_\ell^R(T_0^R = \infty)P_m^L(T_0^L = \infty),$$

where $T_0^R = \inf(n \geqslant 0 : R_n = 0)$ and $T_0^L = \inf(n \geqslant 0 : L_n = 0)$ are the hitting times of zero for both birth and death chains. By a known result on birth-death processes (see §I.4 [69] for instance), and has for any site $x \in \mathbb{Z}$,

$$\lim_{n\to\infty} \mathbb{P}_r^\omega(\alpha_n(x) = 0) \geqslant \frac{\sum_{k=0}^{\ell-1}\prod_{j=0}^{k}\lambda_v(j+1)}{\sum_{k\geqslant 0}\prod_{j=0}^{k}\lambda_v(j+1)}\,\frac{\sum_{k=m+1}^{0}\prod_{j=k}^{0}\lambda_v(j-1)}{\sum_{k\leqslant 0}\prod_{j=k}^{0}\lambda_v(j-1)} \tag{3.3.5}$$

By (3.3.2), this limit tends to 1 when m goes to $-\infty$ and ℓ goes to ∞. With a death rate equal to 1, for all $m < 0 < \ell$, there exists almost surely some time $\bar{t}$ where $\alpha_{\bar{t}}(x) = 0$, for all $x \in \mathbb{Z}$. Coupling the processes $(\alpha_t)_{t\geqslant 0}$ and $(\chi_t)_{t\geqslant 0}$ starting from such times $\bar{t}$, if $\chi_0 \leqslant \alpha_0$ then the dynamics of the coupled process $(\chi_t, \alpha_t)_{t\geqslant 0}$ is given by the following transitions :

transition	rate
$(0,0) \longrightarrow (1,1)$	$\sum_{y:\lvert y-x\rvert=1} \lambda_v(y)\chi(y)$
$(0,0) \longrightarrow (0,1)$	$\sum_{y:\lvert y-x\rvert=1} \lambda_v(y)(\alpha(y)-\chi(y))$
$(1,1) \longrightarrow (0,0)$	$\mathbf{1}\{n_0(x,\alpha) > 0\}$
$(1,1) \longrightarrow (0,1)$	$1 - \mathbf{1}\{n_0(x,\alpha) > 0\}$
$(0,1) \longrightarrow (1,1)$	$\sum_{y:\lvert y-x\rvert=1} \lambda_v(y)\chi(y)$
$(0,1) \longrightarrow (0,0)$	$\mathbf{1}\{n_0(x,\alpha) > 0\}$
$(1,0) \longrightarrow (1,1)$	$\sum_{y:\lvert y-x\rvert=1} \lambda_v(y)\alpha(y)$
$(1,0) \longrightarrow (0,0)$	1

whose dynamics does not reach the second part of the table if $\chi_0 \leqslant \alpha_0$. In other words, the natural order on $\{0,1\}$ is preserved and by [10, Proposition 2.7], $(\alpha_t)_{t\geqslant 0}$ is stochastically larger than $(\chi_t)_{t\geqslant 0}$. Finally one gets,

$$\lim_{t\to\infty} \mathbb{P}_r^\omega(\chi_t(x) = 0) = 1$$

for all $x \in \mathbb{Z}$. □

If the family $\{\omega(k), k \in \mathbb{Z}\}$ is random and i.i.d. then the family $\{\lambda_v(k), k \in \mathbb{Z}\}$ is i.i.d. as well, one deduces the following criterion from Theorem 3.2.1-(2).

Corollary 3.3.1. *The process in random environment $(\chi_t)_{t\geqslant 0}$ dies out if*

$$\mathbb{E}_r^\omega \log \lambda_v(0) < 0.$$

that is, if $\lambda_2 < 1$ and $r > -\log \lambda_2 / \log \lambda_1$.

Proof. By the ergodic theorem,

$$\lim_{k\to\infty} \frac{1}{k} \sum_{j=0}^{k} \log \lambda_v(j) = \mathbb{E}_r^\omega \log \lambda_v(0).$$

Denote by $a_k = \prod_{j=0}^{k} \lambda_v(j)$ the general term of series (3.3.2). Since $\mathbb{E}_r^\omega \log \lambda_v(0) < 0$,

$$\lim_{k\to\infty} \frac{1}{k} \log a_k < -b,$$

for some positive b. That is, $\lim_{k\to\infty} \log a_k < \lim_{k\to\infty} (-bk)$, written $\log a_k \sim_{k\infty} -bk$, and

$$\sum_{k\geqslant 0} a_k = \sum_{k\geqslant 0} \exp(\log a_k) \sim_{k\infty} \sum_{k\geqslant 0} \exp(-bk).$$

Therefore, assumptions (3.3.2) are satisfied as soon as $\mathbb{E}_r^\omega \log \lambda_v(0) < 0$. □

Applying this result to our dynamics given by (3.3.1),

$$\mathbb{E}_r^\omega \log \lambda_v(0) = p \log \lambda_2 + (1-p) \log \lambda_1 < 0 \tag{3.3.6}$$

i.e. $p > \log \lambda_1 / (\log \lambda_1 - \log \lambda_2)$. Since $p < 1$, this implies

$$\lambda_2 < 1. \tag{3.3.7}$$

By (3.2.3) and (3.3.1),

$$\mathbb{E}_r^\omega\big(\log \lambda_v(0)\big) = \frac{r \log \lambda_2 + \log \lambda_1}{r+1},$$

one has under (3.3.7) the following extinction criterion from (3.3.6)

$$r > -\frac{\log \lambda_1}{\log \lambda_2}. \tag{3.3.8}$$

Since we assumed $\lambda_1 > \lambda_c$, the right-hand side is positive and (3.3.8) is an upper bound on the transitional phase with respect to λ_1 and λ_2 for the extinction of the process.

3.3.2 Survival conditions

Applying Theorem 3.2.2, one gets

Theorem 3.3.2. *Assume*

$$\sum_{j\geqslant 0} \mathbb{E}_r^{\omega}\left(\frac{1}{\lambda_v(j)} \prod_{k=1}^{j} \frac{\lambda_v(k)+\lambda_v(k-1)+1}{\lambda_v(k)\lambda_v(k-1)} \right) < \infty.$$

Then the process $(\chi_t)_{t\geqslant 0}$ *in random environment survives.*

The lack of independence in the product of the terms of this series disables us to obtain explicit conditions for survival of the process. Nevertheless, by defining the randomness on the edges rather than on the vertices, meaning that the growth rates emanating from a site k respectively to $k+1$ and to $k-1$ are randomly chosen for each $k \in \mathbb{Z}$, we are able to explicit bounds on r with respect to λ_1 and λ_2.

3.4 Random growth on oriented edges

Let $\{(\rho_e(k), \lambda_e(k)), k \in \mathbb{Z}\}$ be an ergodic, stationary and i.i.d. sequence. For the random growth on oriented edges, given a site $k \in \mathbb{Z}$, a birth from k to $k+1$ occurs at rate $\lambda_e(k+1)$ and independently of a birth from k to $k-1$ occuring at rate $\rho_e(k-1)$. See Figure 3.3

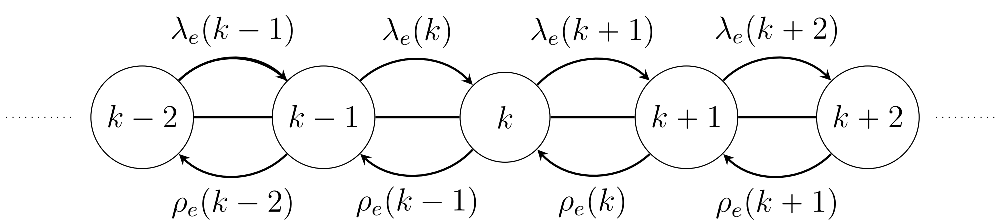

Figure 3.3: Random environment on oriented edges

Suppose both rates are two independent random variables following the same distribution, defined by

$$\lambda_e(k+1) \overset{(d)}{=} \lambda_1(1-\omega(k)) + \lambda_2\omega(k),$$

$$\rho_e(k-1) \overset{(d)}{=} \lambda_1(1-\omega(k)) + \lambda_2\omega(k).$$

Based on the notations provided in Section 3.2, one has

$$\lambda(k) = \lambda_e(k) \text{ and } \rho(k) = \rho_e(k).$$

3.4.1 Extinction conditions

Theorem 3.2.1 permits to obtain the following criteria.

Theorem 3.4.1. *The process in random environment dies out if the two following assertions are satisfied.*

i.

$$\mathbb{E}_r^{\omega}\lambda_e(k) < 1,$$

ii.

$$1 - \mathbb{E}_r^{\omega}\frac{1}{\lambda_e(k)} < \mathbb{E}_r^{\omega}\frac{1}{\lambda_e(k)}\Big(1 - \mathbb{E}_r^{\omega}\lambda_e(k)\Big).$$

that is, if $\lambda_2 < 1$ *and* $r > \dfrac{\lambda_1 - 1}{1 - \lambda_2}$.

Proof. Computing the expectation of the growth rates, conditions on r for the process to die out are given by :

(i) can be rewritten using (3.2.3)

$$\lambda_1(1-p) + \lambda_2 p < 1 \Leftrightarrow r(1-\lambda_2) > \lambda_1 - 1,$$

therefore, as $\lambda_1 - 1 > 0$ since $\lambda_1 > \lambda_c > 1$, one has again

$$\lambda_2 < 1 \tag{3.4.1}$$

and the condition

$$r > \frac{\lambda_1 - 1}{1 - \lambda_2} \tag{3.4.2}$$

On the other hand, (ii) is

$$A(r) = 2r^2\frac{1-\lambda_2}{\lambda_2} + r\Big(\frac{2-\lambda_2}{\lambda_1} + \frac{2-\lambda_1}{\lambda_2} - 2\Big) + 2\frac{1-\lambda_1}{\lambda_1} > 0.$$

The roots of the polynomial are real since its corresponding discriminant Δ is non-negative,

$$\Delta = \frac{1}{\lambda_1^2\lambda_2^2}(\lambda_1 - \lambda_2)^2\Big((\lambda_1 + \lambda_2 - 2)^2 + 4\lambda_1\lambda_2\Big)$$

Roots are therefore given by

$$\delta_{\pm} \quad = \quad \frac{(\lambda_1+\lambda_2-2)(\lambda_1+\lambda_2) \pm (\lambda_1-\lambda_2)\sqrt{(\lambda_1+\lambda_2-2)^2 + 4\lambda_1\lambda_2}}{4\lambda_1(1-\lambda_2)}$$

Consequently, the process in random environment survives as soon as r satisfies

$$(r-\delta_+)(r-\delta_-) > 0$$

Since $\lambda_2 < 1$ and $\lambda_1 > 1$, one has $\delta_+\delta_- = \dfrac{(1-\lambda_1)\lambda_2}{(1-\lambda_2)\lambda_1} < 0$. Both roots δ_- and δ_+ are of opposite sign and $A(r) > 0$ if

$$r > \delta_+, \tag{3.4.3}$$

(because $\delta_- < 0$). Notice condition 3.4.2 implies that $r > \delta_+$. □

3.4.2 Survival conditions

Applying Theorem 3.2.2 to our case where the sequence $\{\rho(k), \lambda(k), k \in \mathbb{Z}\}$ is i.i.d., we get

Theorem 3.4.2. *The process in random environment survives if for all $j \geqslant 0$,*

$$\mathbb{E}_r^\omega\left(\frac{1}{\lambda_e(j+1)}\right)\left(\mathbb{E}_r^\omega \frac{\lambda_e(k)+\rho_e(k-1)+1}{\lambda_e(k)\rho_e(k-1)}\right)^j < 1$$

that is, if $\lambda_2 < 1+\sqrt{2} < \lambda_1$ and $r < \dfrac{\lambda_2\left(\lambda_1-\sqrt{2}-1\right)}{\lambda_1\left(\lambda_2-\sqrt{2}-1\right)}$.

Proof. The (geometric) series converges as soon as

$$\mathbb{E}_r^\omega \frac{\lambda_e(0)+\rho_e(0)+1}{\lambda_e(0)\rho_e(0)} < 1,$$

that is, if

$$2\mathbb{E}_r^\omega \frac{1}{\lambda_e(k)} + \mathbb{E}_r^\omega \frac{1}{\lambda_e(k)\rho_e(k-1)} = \frac{2\lambda_1+1}{\lambda_1^2}(1-p)^2 + 2\frac{\lambda_1+\lambda_2+1}{\lambda_1\lambda_2}p(1-p) + \frac{2\lambda_2+1}{\lambda_2^2}p^2$$

smaller than 1 i.e. using (3.2.3) if,

$$A(r) := r^2\left[\lambda_1^2(2\lambda_2+1) - \lambda_1^2\lambda_2^2\right] + r\lambda_1\lambda_2\left[2(\lambda_1+\lambda_2+1) - 2\lambda_1\lambda_2\right] + \left[\lambda_2^2(2\lambda_1+1) - \lambda_1^2\lambda_2^2\right] < 0. \quad (3.4.4)$$

The associated discriminant is $\Delta = 8\lambda_1^2\lambda_2^2(\lambda_1-\lambda_2)^2$. And the roots of $A(r)$ are

$$\delta_-^A = \frac{-\lambda_2\left(\lambda_1+\sqrt{2}-1\right)}{\lambda_1\left(\lambda_2+\sqrt{2}-1\right)} < 0$$

and

$$\delta_+^A = \frac{-\lambda_2\left(\lambda_1-\sqrt{2}-1\right)}{\lambda_1\left(\lambda_2-\sqrt{2}-1\right)},$$

which is positive if

$$\lambda_1 > 1+\sqrt{2}, \quad (3.4.5)$$

and $\lambda_2 < 1+\sqrt{2}$, this last condition is cleared by the assumption (3.2.2), as $\lambda_2 < \lambda_c \leqslant 1+\sqrt{2}$. In this case, the process survives if r is such that

$$r < \delta_+^A. \quad (3.4.6)$$

□

3.5 Numerical bounds on the transitional phase

3.5.1 Back to the basic contact process

Assume $r = 0$, then for all $x \in \mathbb{Z}$, $\omega(x) = 0$ a.s. and $\lambda_e(x) = \rho_e(x) = \lambda_1$. We thus recover the one-dimensional basic contact process with growth rate λ_1. In this case, our estimates lead to the following bound for λ_c.

Corollary 3.5.1. *For the one-dimensional basic contact process,*

$$\lambda_c \leqslant 1 + \sqrt{2}.$$

Proof. According to (3.4.4) in the proof of Theorem 3.4.2, the process survives if $\lambda_1^2 - 2\lambda_1 - 1 > 0$, that is, if

$$\lambda_1 > 1 + \sqrt{2}.$$

□

Recall on $\mathbb{Z}$, $\lambda_c \in [1.539, 1.942]$. This bound is quite rough but its advantage is that we derived it simply. Consequently, one first deduces a bound on the critical value λ_c of the one-dimensional basic contact process : $\lambda_c \leqslant 1 + \sqrt{2} \simeq 2.41$.

3.5.2 The phase transition

From results obtained in the previous section, one gets the following numerical bounds for a phase transition. By choosing parameters λ_1 and λ_2 satisfying (3.4.1), condition (3.4.2) from Theorem 3.4.1 gives us lower bounds on the phase transition. Moreover, by choosing parameters λ_2 and λ_1 satisfying (3.4.5), condition (3.4.6) from Theorem 3.4.2 gives us upper bounds.

λ_1	λ_2	transitional phase
1000	0.2	$[0.07, 1249]$
100	0.2	$[0.07, 124]$
10	0.2	$[0.044, 11.25]$
2	0.2	$[0, 1.25]$

λ_1	λ_2	transitional phase
1000	0.8	$[0.49, 4995)$
100	0.8	$[0.48, 495)$
10	0.8	$[0.36, 45]$
2	0.8	$(0, 5]$

λ_1	λ_2	transitional phase
1000	1.4	$[1.37, \infty)$
100	1.4	$[1.34, \infty)$
10	1.4	$[1.04, \infty)$
2	1.4	$\mathbb{R}_+$

Remark that the necessary condition $\lambda_2 < 1$ disables us to conclude to an upper bound for values of λ_2. In a similar way, condition (3.4.5) of Theorem 3.4.2 imposes λ_1 to be larger than $1 + \sqrt{2}$, disabling us to find an explicit lower bound on the transitional phase in such cases.

4

Hydrodynamic limit on the d-dimensional torus

Contents

4.1 Introduction

In this chapter, one derives the hydrodynamic limit on the d-dimensional torus of the asymmetric multitype contact process defined in Chapter 1.

The work here is is based on the entropy method due to M. Z. Guo, G. C. Papanicolaou and S. R. S. Varadhan [37] to prove the hydrodynamic behaviour of a large class of interacting particle systems through the investigation of the time-evolution of the entropy and arguments by C. Kipnis, S. Olla and S.R.S Varadhan [43], using martingales techniques.

This chapter is a preliminary to the next one, it introduces many involved quantities and we detail here classical computations that appear in both chapters. It is organized as follows. We begin by describing the model and the main result in Section 4.2, which is subsequently proved in Section 4.3, while classical proofs concerning the block estimates are proved in Section 4.4.

In the Appendix 4.A, we deal with a construction of an auxiliary process, a trick introduced by M. Mourragui [63], in case of unbounded rates. Whereafter, we expose some lengthy computations surrounding the reference measure (Appendix 4.B) and reminders on the Skorohod topology (Appendix 4.D).

4.2 Notations and Results

Let $\mathbb{T}_N^d = (\mathbb{Z}/N\mathbb{Z})^d$ be the $d-$dimensional discrete microscopic torus $\{0, ..., N-1\}^d$ and $\mathbb{T}^d = (\mathbb{R}/\mathbb{Z})^d$ be the corresponding macroscopic torus $[0,1)^d$.

4.2.1 The model

Define $E_N = \{0,1,2,3\}^{\mathbb{T}_N^d}$. The model we investigate is a reaction-diffusion process $(\eta_t)_{t\geqslant 0}$ given by the generator

$$\mathfrak{L}_N := \mathfrak{L}_{N,R,D} = N^2\mathcal{L}_N^D + \mathcal{L}_N^R, \tag{4.2.1}$$

where $N^2\mathcal{L}_N^D$ stands for the generator of a rapid-stirring process, defined for any function f on E_N by

$$N^2\mathcal{L}_N^D f(\eta) = N^2 \sum_{\substack{x,y\in\mathbb{T}_N^d \\ \|x-y\|=1}} \Big(f(\eta^{x,y}) - f(\eta)\Big), \tag{4.2.2}$$

here, $\|x\| = \max_{1\leqslant j\leqslant d} |x_j|$ denotes the max norm for $x \in \mathbb{Z}^d$, and $\eta \in E_N$, $\eta^{x,y}$ is the configuration obtained from η by exchanging the occupation variables $\eta(x)$ and $\eta(y)$ of two neighbouring sites $x, y \in \mathbb{T}_N^d$, that is,

$$\eta^{x,y}(z) = \begin{cases} \eta(x) \text{ if } z = y, \\ \eta(y) \text{ if } z = x, \\ \eta(z) \text{ if } z \neq x, y, \end{cases}$$

and $\mathcal{L}_N^R$ is the generator of the asymmetric CP-DRE defined in Chapter 2, which is given for any cylinder function f on E_N by

$$\mathcal{L}_N^R f(\eta) = \sum_{i=0}^{3} c(x,\eta,i)\Big(f(\eta_x^i) - f(\eta)\Big) \tag{4.2.3}$$

with $\eta \in E_N$, η_x^i is the configuration obtained from η by flipping the state of site x into the state $i \in \{0,1,2,3\}$, that is,

$$\eta_x^i(z) = \begin{cases} i \text{ if } z = x, \\ \eta(z) \text{ if } z \neq x. \end{cases}$$

while the rate function c is defined by

$$c(x,\eta,0) = 1 \text{ if } \eta(x) \in \{1,2\},$$

$$c(x,\eta,1) = \begin{cases} \beta(x,\eta) := \lambda_1 \sum\limits_{\substack{y\in\mathbb{T}_N^d \\ \|y-x\|=1}} \eta_1(y) + \lambda_2 \sum\limits_{\substack{y\in\mathbb{T}_N^d \\ \|y-x\|=1}} \eta_3(y) \text{ if } \eta(x)=0, \\ 1 \text{ if } \eta(x) = 3, \end{cases} \tag{4.2.4}$$

$$c(x,\eta,2) = \begin{cases} r \text{ if } \eta(x) = 0, \\ 1 \text{ if } \eta(x) = 3, \end{cases}$$

$$c(x,\eta,3) = \ r \text{ if } \eta(x) = 1.$$

Since the conserved quantities for the generator $\mathcal{L}_N^D$ concern the total number of particles of each type $i \in \{1,2,3\}$, one defines the product measure $\overline{\nu}_{\widehat{\psi}}^N$ on E_N by

$$\overline{\nu}_{\widehat{\psi}}^N(\eta) := \prod_{x\in\mathbb{T}_N^d} \frac{1}{Z_{\widehat{\psi}}} \exp\left(\sum_{i=0}^{3} [\psi_i \mathbf{1}\{\eta(x)=i\}] \right) \tag{4.2.5}$$

where $Z_{\widehat{\psi}} = \sum\limits_{i=0}^{3} \exp(\psi_i)$ is the normalization constant, for $\widehat{\psi} = (\psi_0,\psi_1,\psi_2,\psi_3)$ such that $\psi_0,\psi_1,\psi_2,\psi_3 \in \mathbb{R}$ are parameters. Because of a high use of indicator functions, we shall simplify the notation by

$$\eta_i(x) = \mathbf{1}\{\eta(x) = i\},$$

for $x \in \mathbb{T}_N^d$ and $i = 1,2,3$.

As usual, we parametrize the measure by the conserved quantities (see for instance R. Marra and M. Mourragui [61]). By a change of variables (see Appendix 4.B for details), given parameters ρ_1,ρ_2,ρ_3 such that $\rho_i \geqslant 0$ and $\rho_1+\rho_2+\rho_3 \leqslant 1$, one defines the product measure for $\widehat{\rho} = (\rho_1,\rho_2,\rho_3)$ by

$$\nu_{\widehat{\rho}}^N(\cdot) = \overline{\nu}_{\Psi(\rho_1,\rho_2,\rho_3)}^N(\cdot) \quad \text{and} \quad \rho_0 = 1-\rho_1-\rho_2-\rho_3. \tag{4.2.6}$$

where Ψ is a bijection from $\mathbb{R}_+^3$ to $(0,1)^3$ given by (4.B.3). The measures $\{\nu_{\widehat{\rho}}^N, \widehat{\rho} \in [0,1]^3\}$ are invariant [see Lemma 4.B.1] with respect to the rapid-stirring process with generator $N^2\mathcal{L}_N^D$, and they are parametrized by the densities :

$$\begin{cases} \mathbb{E}_{\nu_{\widehat{\rho}}}^N[\eta_k(x)] = \nu_{\widehat{\rho}}^N(\eta(x)=k) = \rho_k,\ 1 \leqslant k \leqslant 3, \\ \nu_{\widehat{\rho}}^N(\eta(x)=0) = 1-\rho_1-\rho_2-\rho_3. \end{cases}$$

For any function ϕ on E_N, denote by $\widetilde{\phi}(\widehat{\rho})$ the expectation of ϕ with respect to $\nu_{\widehat{\rho}}^N$:

$$\widetilde{\phi}(\widehat{\rho}) = \mathbb{E}_{\nu_{\widehat{\rho}}}^N[\phi(\eta)]. \tag{4.2.7}$$

To do changes of variables, it will be more convenient to write the measures as follows :

$$\nu_{\hat{\rho}}^N(\eta) = \exp\Big\{\sum_{j=0}^{3}\sum_{x\in\mathbb{T}_N^d}\varrho_j\eta_j(x)\Big\} \tag{4.2.8}$$

$$\text{with} \qquad \varrho_j = \log\rho_j \tag{4.2.9}$$

Since conserved quantities are densities of three types of particles, we need to work with three dimensional vectors whose i-th component is associated to the type i. These vectors will be distinguished with a hat. For any configuration η, define the empirical measure of type i on E_N by

$$\pi^{N,i}(\eta) = \frac{1}{N^d}\sum_{x\in\mathbb{T}_N^d}\eta_i(x)\delta_{\frac{x}{N}}, \tag{4.2.10}$$

where $\delta_{\frac{x}{N}}(dy)$ stands for the Dirac measure at x/N. And note for $(\eta_s)_{s\in[0,T]}$,

$$\widehat{\pi}_t^N(\eta) := (\pi_t^{N,1},\pi_t^{N,2},\pi_t^{N,3})(\eta), \tag{4.2.11}$$

where $\pi_t^{N,i}(\eta) = \pi^{N,i}(\eta_t)$. Let $\mathcal{C}^{n,m}([0,T]\times\mathbb{T}^d;\mathbb{R})$ be the set of functions n times continuously differentiable in time and m times continuously differentiable in space. For any function $G_i \in \mathcal{C}^{1,2}([0,T]\times\mathbb{T}^d;\mathbb{R})$, denote the integral of $G_{i,t}$ with respect to $\pi_t^{N,i}$ by

$$\langle\pi_t^{N,i}, G_{i,t}\rangle = \frac{1}{N^d}\sum_{x\in\mathbb{T}_N^d}G_{i,t}(x/N)\eta_i(x). \tag{4.2.12}$$

For any function $\widehat{G}_t = (G_{1,t},G_{2,t},G_{3,t}) \in \mathcal{C}^{1,2}([0,T]\times\mathbb{T}^d;\mathbb{R}^3)$, define the integral of $\widehat{G}_t$ with respect to $\widehat{\pi}_t^N$ by

$$\langle\widehat{\pi}_t,\widehat{G}_t\rangle = \sum_{i=1}^{3}\langle\pi_t^{N,i},G_{i,t}\rangle.$$

4.2.2 Hydrodynamics for the reaction-diffusion process

Let $\mathcal{M}_+^1$ be the subset of $\mathcal{M}$ of all positive measures absolutely continuous with respect to the Lebesgue measure with positive density bounded by 1 :

$$\mathcal{M}_+^1 = \left\{\pi\in\mathcal{M} : \pi(du) = \rho(u)du \quad\text{and}\quad 0\leqslant\rho(u)\leqslant 1 \ \text{ a.e.}\right\}.$$

Fix $T>0$. Let $D([0,T],(\mathcal{M}_+^1)^3)$ be the set of right-continuous with left limits trajectories with values in $(\mathcal{M}_+^1)^3$, endowed with the Skorohod topology and equipped with its Borel $\sigma-$ algebra.

For any probability measure μ on E_N, denote by $\mathbb{P}_{\mu^N}^N$ the probability measure on $D([0,T],E_N)$ of the process $(\eta_t)_{t\in[0,T]}$ with generator $\mathfrak{L}_N$ and by $\mathbb{E}_{\mu^N}^N$ the corresponding expectation. Consider $Q_\mu^N = \mathbb{P}_{\mu^N}^N\circ(\widehat{\pi}^N)^{-1}$ the law of the process $\big(\widehat{\pi}_t^N(\eta_t)\big)_{t\in[0,T]}$.

Fix $T > 0$. A sequence of probability measures is associated to a density profile $\widehat{\gamma} = (\gamma_1, \gamma_2, \gamma_3) : \mathbb{T}^d \to [0,1]^3$ if for any $\delta > 0$ and any function $\widehat{G} \in \mathcal{C}^1(\mathbb{T}^d, \mathbb{R}^3)$,

$$\lim_{N\to\infty} \mu^N \Big\{ \Big| \langle \widehat{\pi}_N(\eta), \widehat{G}(\cdot)\rangle - \langle \widehat{\gamma}(\cdot), \widehat{G}(\cdot)\rangle \Big| > \delta \Big\} = 0\,, \tag{4.2.13}$$

Denote by $\widehat{\rho} = (\rho_1, \rho_2, \rho_3) : [0,T] \times \mathbb{T}^d \to [0,1]^3$ a typical macroscopic trajectory. We shall show that the macroscopic time-evolution of empirical density $\widehat{\pi}^N$ is given by a reaction-diffusion system

$$\begin{cases} \partial_t \widehat{\rho} & = & \Delta \widehat{\rho} + \widehat{\mathfrak{R}}(\widehat{\rho}) & \text{in} & \mathbb{T}^d \times (0,T), \\ \widehat{\rho}_0(\cdot) & = & \widehat{\gamma}(\cdot) & \text{in} & \mathbb{T}^d, \end{cases} \tag{4.2.14}$$

where $\widehat{\mathfrak{R}} = (\mathfrak{R}_1, \mathfrak{R}_2, \mathfrak{R}_3) : [0,1]^3 \to \mathbb{R}^3$ is defined by

$$\begin{cases} \mathfrak{R}_1(\rho_1,\rho_2,\rho_3) & = & 2d(\lambda_1\rho_1 + \lambda_2\rho_3)\rho_0 + \rho_3 - \rho_1(r+1)\,, \\ \mathfrak{R}_2(\rho_1,\rho_2,\rho_3) & = & r\rho_0 + \rho_3 - \rho_2\,, \\ \mathfrak{R}_3(\rho_1,\rho_2,\rho_3) & = & r\rho_1 - 2\rho_3\,, \end{cases} \tag{4.2.15}$$

with $\rho_0 = 1 - \rho_1 - \rho_2 - \rho_3$. A weak solution $\widehat{\rho}(\cdot,\cdot) : [0,T] \times \mathbb{T}^d \to \mathbb{R}^3$ of (4.2.14) satisfies the following assertions :

(S1) For any $i \in \{1,2,3\}$, $\rho_i \in L^2\big([0,T] \times \mathbb{T}^d\big)$.

(S2) For any function $\widehat{G}(t,u) = \widehat{G}_t(u) = (G_{1,t}(u), G_{2,t}(u), G_{3,t}(u))$ in $\mathcal{C}^{1,2}\big([0,T] \times \mathbb{T}^d; \mathbb{R}^3\big)$, one has

$$\langle \widehat{\rho}_T, \widehat{G}_T\rangle - \langle \widehat{\rho}_0, \widehat{G}_0\rangle = \int_0^T ds\, \langle \widehat{\rho}_s, (\partial_s + \Delta)\widehat{G}_s\rangle + \int_0^T ds\, \langle \widehat{\mathfrak{R}}(\widehat{\rho}_s), \widehat{G}_s\rangle, \tag{4.2.16}$$

here for $\widehat{G}, \widehat{H} \in \mathcal{C}^{1,2}([0,T]\times\mathbb{T}^d;\mathbb{R}^3)$, $\langle \widehat{G}(\cdot), \widehat{H}(\cdot)\rangle$ stands for the usual inner product of $L^2(\mathbb{T}^d)$: $\langle \widehat{G}, \widehat{H}\rangle = \sum_{i=1}^3 \int_{\mathbb{T}^d} G_i(u)H_i(u)du$.

The rest of this chapter is devoted to prove the following result.

Theorem 4.2.1. *Let $\widehat{\gamma} : \mathbb{T}^d \to [0,1]^3$ be an initial continuous profile and $(\mu^N)_{N\geqslant 1}$ be a sequence of probability measures with μ^N a probability measure on E_N for each N associated to $\widehat{\gamma}$. The sequence of random measures $(\widehat{\pi}^N_t)_{N\geqslant 1}$ converges weakly in probability as N goes to infinity to the absolutely continuous measure $\widehat{\pi}_t(du) = \widehat{\rho}(t,u)du$ whose density $\widehat{\rho}(t,u) = (\rho_1,\rho_2,\rho_2)(t,u)$ is the unique weak solution of the reaction-diffusion system* (4.2.14)*. That is, for any $t \in [0,T]$, any $\delta > 0$ and any function $\widehat{G} \in \mathcal{C}(\mathbb{T}^d, \mathbb{R}^3)$*

$$\lim_{N\to\infty} \mathbb{P}^N_{\mu^N} \Big\{ \Big| \langle \widehat{\pi}_N(\eta_t), \widehat{G}(\cdot)\rangle - \langle \widehat{\rho}_t(\cdot), \widehat{G}(\cdot)\rangle \Big| > \delta \Big\} = 0\,.$$

4.3 The hydrodynamic limit

For any function $\widehat{G} = (G_1, G_2, G_3) \in \mathcal{C}^{1,2}([0,T] \times \mathbb{T}^d; \mathbb{R}^3)$, by Dynkin's formula

$$M_t^{N,i} = \langle \pi_t^{N,i}, G_{i,t} \rangle - \langle \pi_0^{N,i}, G_{i,0} \rangle - \int_0^t \mathfrak{L}_N \langle \pi_s^{N,i}, G_{i,s} \rangle ds - \int_0^t \langle \pi_s^{N,i}, \partial_s G_{i,s} \rangle ds \tag{4.3.1}$$

is a $\mathbb{Q}_{\mu^N}^N-$martingale with respect to the $\sigma-$algebra $\mathcal{F}_t = \sigma(\eta_s,\ s \leqslant t)$.

To derive the hydrodynamic behaviour of the reaction-diffusion process, one needs to prove that the above martingale vanishes as N goes to infinity. To this purpose, apply the generator $\mathfrak{L}_N$ to the function $\eta \to \eta_i(x)$ so that the integral part of $M_t^{N,i}$ is depicted as follows.

$$\begin{aligned}N^2 \mathcal{L}_N^D \langle \pi_t^{N,i}, G_{i,t} \rangle &= \frac{N^2}{N^d} \sum_{x \in \mathbb{T}_N^d} \sum_{j=1}^d G_{i,t}(x/N) \Big(\eta_{i,t}(x+e_j) + \eta_{i,t}(x-e_j) - 2\eta_{i,t}(x) \Big) \\ &= \langle \pi_t^{N,i}, \Delta_N G_{i,t}(\cdot) \rangle,\end{aligned}$$

where $\Delta_N G_{i,t}(x/N) = N^2 \sum_{j=1}^d (G_{i,t}((x+e_j)/N) + G_{i,t}((x-e_j)/N) - 2G_{i,t}(x/N))$ is the discrete laplacian in dimension d and $(e_1, ..., e_d)$ is the canonical basis of $\mathbb{R}^d$. And computing $\mathcal{L}_N^R \eta_i(x)$ for each i gives

$$\mathcal{L}_N^R \eta_1(x) = \sum_{x \in \mathbb{T}_N^d} \left(\lambda_1 \sum_{y : \|y-x\|=1} \eta_1(y) + \lambda_2 \sum_{y : \|y-x\|=1} \eta_3(y) \right) \eta_0(x) + \eta_3(x) - (r+1)\eta_1(x), \tag{4.3.2}$$

$$\mathcal{L}_N^R \eta_2(x) = r\eta_2(x) + \eta_3(x) - \eta_2(x), \tag{4.3.3}$$

$$\mathcal{L}_N^R \eta_3(x) = r\eta_1(x) - 2\eta_3(x), \tag{4.3.4}$$

so that we deduce

$$\begin{aligned}&\mathcal{L}_N^R \langle \pi_t^{N,1}, G_{1,t} \rangle = \frac{1}{N^d} \sum_{x \in \mathbb{T}_N^d} G_{1,t}(x/N) \left(\lambda_1 \sum_{y : \|y-x\|=1} \eta_{1,t}(y) + \lambda_2 \sum_{y : \|y-x\|=1} \eta_{3,t}(y) \right) \eta_{0,t}(x) \\ &\quad + \langle \pi_t^{N,3}, G_{1,t} \rangle - (r+1) \langle \pi_t^{N,1}, G_{1,t} \rangle, \\ &\mathcal{L}_N^R \langle \pi_t^{N,2}, G_{2,t} \rangle = r \langle \pi_t^{N,0}, G_{2,t} \rangle + \langle \pi_t^{N,3}, G_{2,t} \rangle - \langle \pi_t^{N,2}, G_{2,t} \rangle \\ &, \mathcal{L}_N^R \langle \pi_t^{N,3}, G_{3,t} \rangle = r \langle \pi_t^{N,1}, G_{3,t} \rangle - 2 \langle \pi_t^{N,3}, G_{3,t} \rangle.\end{aligned} \tag{4.3.5}$$

Thus, to close the equations we need to replace the local function of η which is the term in $\mathcal{L}_N^R \eta_1(x)$ by a functional of the empirical densities given by $\widehat{\pi}_t^N$ defined in (4.2.11). This is the purpose of the replacement lemma and the blocks estimates, exposed in Sections 4.3.2 and 4.4.

Next, we need to characterize all the limit points of the sequence $(\mathbb{Q}^N_{\mu^N})_{N\geqslant 1}$: their existence comes from by the tightness of the sequence of measures, it is proved in Section 4.3.1, then, the identification and uniqueness of the limit points as weak solutions of (4.2.14) conclude the proof in Sections 4.3.3 and 4.3.4.

4.3.1 Tightness

Existence of limit points is guaranteed by the following lemma.

Lemma 4.3.1 (Tightness). *The sequence $(Q^N_\mu)_{N\geqslant 1}$ is tight and all its limit points Q^*_μ satisfy*

$$Q^*_\mu\Big(\widehat{\pi} : 0 \leqslant \widehat{\pi}_t(u) \leqslant 1, \widehat{\pi}_t(u) = \widehat{\pi}_t(u)du,\ t \in [0,T]\Big) = 1. \tag{4.3.6}$$

Proof. By Proposition 4.D.4, it is enough to show tightness for the real-valued process $\{\langle\widehat{\pi}_t, \widehat{G}\rangle, t \in [0,T]\}$ for all functions $\widehat{G} \in \mathcal{C}^2(\mathbb{T}^d;\mathbb{R}^3)$. By Prohorov's theorem 4.D.1, to get the tightness of $\{\langle\widehat{\pi}_t, \widehat{G}\rangle, t \in [0,T]\}$ in $D([0,T],\mathbb{R}^3)$ with the uniform topology, one needs to check the two following assertions :

(i) boundedness :

$$\lim_{m\to\infty}\lim_{N\to\infty} Q^N_{\mu^N}\Big(\sup_{t\in[0,T]}|\langle\widehat{\pi}_t, \widehat{G}\rangle| \geqslant m\Big) = 0. \tag{4.3.7}$$

(ii) equicontinuity :

$$\lim_{\delta\to 0}\overline{\lim_{N\to\infty}}\, Q^N_{\mu^N}\Big(\sup_{|t-s|\leqslant\delta}|\langle\widehat{\pi}_t, \widehat{G}\rangle - \langle\widehat{\pi}_s, \widehat{G}\rangle| > \epsilon\Big) = 0,\ \text{for any } \epsilon > 0. \tag{4.3.8}$$

The limit (4.3.7) is immediate since for each $t \in [0,T]$ and $1 \leqslant i \leqslant 3$, the total mass of $\pi^{N,i}_t$ is bounded by 1. To prove (4.3.8), it is enough to show for the martingale $M^{N,i}_t$ defined in (4.3.1) that

$$\lim_{\delta\to 0}\lim_{N\to\infty} Q^N_{\mu^N}\Big(\sup_{|t-s|\leqslant\delta}|M^{N,i}_t - M^{N,i}_s| > \epsilon\Big) = 0, \quad \text{for any } \epsilon > 0 \tag{4.3.9}$$

and

$$\lim_{\delta\to 0}\lim_{N\to\infty} Q^N_{\mu^N}\Big(\sup_{0\leqslant t-s\leqslant\delta}\Big|\int_s^t \mathfrak{L}_N\langle\pi^{N,i}_r, G_i\rangle dr\Big| > \epsilon\Big) = 0, \quad \text{for any } \epsilon > 0. \tag{4.3.10}$$

To prove (4.3.9), one shows the quadratic variation $\langle M^{N,i}\rangle_t$ of the martingale $M^{N,i}_t$ converges to zero as N goes to ∞. Note that since $\widehat{G}$ is not time-dependent, the time derivative of $\widehat{G}$ is null in the expression (4.3.1). By the Doob-Meyer decomposition,

$$\langle M^{N,i}\rangle_t = \int_0^t \Big\{\mathfrak{L}_N\langle\pi^{N,i}_s, G_i\rangle^2 - 2\langle\pi^{N,i}_s, G_i\rangle\mathfrak{L}_N\langle\pi^{N,i}_s, G_i\rangle\Big\}ds. \tag{4.3.11}$$

We postpone the detailed computations to Appendix 4.C. By Lemma 4.C.1, one has

$$N^2 \int_0^t \Big\{ \mathcal{L}_N^D \langle \pi_s^{N,i}, G_i \rangle^2 - 2\langle \pi_s^{N,i}, G_i \rangle \mathcal{L}_N^D \langle \pi_s^{N,i}, G_i \rangle \Big\} ds \leqslant C(G) t N^{-d}$$

$$\int_0^t \Big\{ \mathcal{L}_N^R \langle \pi_s^{N,i}, G_i \rangle^2 - 2\langle \pi_s^{N,i}, G_i \rangle \mathcal{L}_N^R \langle \pi_s^{N,i}, G_i \rangle \Big\} ds \leqslant C(\lambda_1, \lambda_2, r) t \|G_i\|_2^2 N^{-d}$$

where $C(\lambda_1, \lambda_2, r)$ stands for the supremum of the bounded rates since all involved rates in factor in (4.C.6) are positive. Therefore, combining both estimates,

$$\langle M^{N,i} \rangle_t \leqslant \frac{\big(C(\lambda_1, \lambda_2, r)\|G_i\|_2^2 + C(G)\big)t}{N^d}, \tag{4.3.12}$$

which converges to zero as $N \to \infty$, one deduces (4.3.9) by using Doob's martingale inequality.

To prove (4.3.10), on one hand,

$$\big| N^2 \mathcal{L}_N^D \langle \pi_t^{N,i}, G_i \rangle \big| = \big| \langle \pi_t^{N,i}, \Delta_N G_i \rangle \big| \leqslant \|\Delta G_i\|_1,$$

where ΔG stands for the Laplace operator $\Delta G = \sum_{j=1}^{d} \partial_{e_j}^2 G$ when ∂_{e_j} is the first derivative in the j-th direction. On the other hand, since all rates (4.2.4) are bounded, by (4.3.5),

$$\big| \mathcal{L}_N^R \langle \pi_t^{N,i}, G_i \rangle \big| \leqslant C(\lambda_1, \lambda_2, r) \|G_i\|_1.$$

To show that $\widehat{\pi}_t$ is absolutely continuous, remark that for any function $\widehat{G} \in \mathcal{C}(\mathbb{T}^d, \mathbb{R}^3)$,

$$\sup_{t \in [0,T]} |\langle \widehat{\pi}_t^N, \widehat{G} \rangle| \leqslant \|\widehat{G}\|_\infty.$$

Hence, since $\widehat{\pi}_\cdot \mapsto \sup_{t \in [0,T]} |\langle \widehat{\pi}_t, \widehat{G} \rangle|$ is continuous with respect to the Skorohod topology, any limit point satisfies by Portmanteau theorem,

$$\sup_{t \in [0,T]} |\langle \widehat{\pi}_t, \widehat{G} \rangle| \leqslant \|\widehat{G}\|_1$$

that is, any limit point is supported on trajectories such that $\widehat{\pi}_t$ is absolutely continuous with respect to the Lebesgue measure for all $t \in [0, T]$. □

4.3.2 Replacement lemma

For any positive integer k and $x \in \mathbb{T}_N^d$, denote by $\eta_i^k(t,x)$ the empirical density of type$-i$ particles given by

$$\eta_i^k(t,x) = \frac{1}{(2k+1)^d} \sum_{\substack{y \in \mathbb{T}_N^d \\ \|y-x\| \leqslant k}} \eta_{i,t}(y), \tag{4.3.13}$$

and define the vector $(\widehat{\eta}_t^k)(x) = (\eta_1^k, \eta_2^k, \eta_3^k)(t,x)$. We now deal with replacing local functions of η by functions of the empirical density within a macroscopic box, in other words, for any cylinder function ϕ and the function $\widetilde{\phi}(\cdot)$ defined by (4.2.7), one shows for any continuous function G and $\epsilon > 0$ the following replacement lemma,

Proposition 4.3.1. *For all $a > 0$,*

$$\varlimsup_{\epsilon\to 0} \varlimsup_{N\to\infty} \frac{1}{N^d} \log \mathbb{P}^N_{\mu^N} \left(\frac{1}{N^d} \int \sum_{x\in T_N^d} \tau_x V_{\epsilon N}(\eta_t) dt \geqslant a \right) = -\infty,$$

where $V_k(\eta) := \left| \dfrac{1}{(2k+1)^d} \sum_{\|y\|\leqslant k} \tau_y \phi(\eta) - \widetilde{\phi}(\widehat{\eta}^k(0)) \right|.$

Proof. For any $\gamma > 0$, by Markov's inequality,

$$\mathbb{P}^N_{\nu^N_{\widehat{\rho}}} \left(\frac{1}{N^d} \int \sum_{x\in \mathbb{T}_N^d} \tau_x V_{\epsilon N}(\eta_t) dt \geqslant a \right) \leqslant \exp(-\gamma N^d a) \cdot \mathbb{E}^N_{\nu^N_{\widehat{\rho}}} \left[\exp\left(\gamma \int_0^T \sum_{x\in \mathbb{T}_N^d} \tau_x V_{\epsilon N}(\eta_t) dt \right) \right].$$

Introduce in $\mathbb{L}^2(\nu^N_{\widehat{\rho}})$ the operator

$$\mathcal{A}_{N,\gamma} := \frac{1}{2}(\mathfrak{L}_N + (\mathfrak{L}_N)^\star) + \gamma \sum_{x\in \mathbb{T}_N^d} \tau_x V_{\epsilon N} \tag{4.3.14}$$

where $(\mathfrak{L}_N)^\star$ is the adjoint of $\mathfrak{L}_N$ in $\mathbb{L}^2(\nu^N_{\widehat{\rho}})$.

Fix $T > 0$, by Feynman-Kac formula (see [42, Appendix 1.7]), for all $t \in [0,T]$, the unique solution of the differential equation

$$\begin{cases} \partial_t u(t,\eta) = \dfrac{1}{2}(\mathfrak{L}_N + (\mathfrak{L}_N)^*) u(t,\eta) + \gamma \sum\limits_{x\in \mathbb{T}_N^d} \tau_x V_{\epsilon N} u(t,\eta) \\ u(0,\eta) = 1 \end{cases} \tag{4.3.15}$$

is given by $u(t,\eta) = \mathbb{E}^N_{\nu^N_{\widehat{\rho}}} \left[u(0,\eta) \exp\left(\int_0^T \gamma \sum_{x\in \mathbb{T}_N^d} \tau_x V_{\epsilon N}(\eta_s) ds \right) \right].$

By the spectral decomposition of the auto-adjoint operator $\mathcal{A}_{N,\gamma}$,

$$\lambda_{\epsilon N}(\gamma) = \sup_{\|u\|=1} \langle \mathcal{A}_{N,\gamma} u, u \rangle \tag{4.3.16}$$

is the largest eigenvalue of the operator $\mathcal{A}_{N,\gamma}$, so that

$$\mathbb{E}^N_{\nu^N_{\widehat{\rho}}} \left[\exp\left(\int_0^T \gamma \sum_{x\in \mathbb{T}_N^d} \tau_x V_{\epsilon N}(\eta_s) ds \right) \right] \leqslant \exp(T \lambda_{\epsilon N}(\gamma))$$

hence, $\dfrac{1}{N^d}\log \mathbb{P}^N_{\nu^N_{\hat\rho}}\left(\dfrac{1}{N^d}\displaystyle\int_0^T\sum_{x\in\mathbb{T}^d_N}\tau_x V_{\epsilon N}(\eta_s)ds \geqslant a\right)\leqslant \dfrac{1}{2N^d}T\lambda_{\epsilon N}(\gamma)-\gamma a.$

It thus remains to show for all $\gamma>0$,

$$\overline{\lim_{\epsilon\to 0}}\,\overline{\lim_{N\to\infty}}\,\frac{\lambda_{\epsilon N}(\gamma)}{N^d}=0, \tag{4.3.17}$$

in which case one would have for all $\gamma>0$,

$$\overline{\lim_{\epsilon\to 0}}\,\overline{\lim_{N\to\infty}}\,\frac{1}{N^d}\log \mathbb{P}^N_{\nu^N_{\hat\rho}}\left(\frac{1}{N^d}\int_0^T\sum_{x\in\mathbb{T}^d_N}\tau_x V_{\epsilon N}(\eta_s)ds \geqslant a\right)\leqslant -\gamma a$$

and conclusion will follow by letting γ go to infinity. By Rayleigh-Ritz variational formula,

$$\begin{aligned}\lambda_{\epsilon N}(\gamma) &= \sup_{\substack{f^N\in\mathbb{L}^2(\nu^N_{\hat\rho})\\ \|f^N\|_{L^2}=1}}\left(\int\gamma\sum_{x\in\mathbb{T}^d_N}\tau_x V_{\epsilon N}(\eta)(f^N)^2(\eta)d\nu^N_{\hat\rho}(\eta)+\langle\mathfrak{L}_N f^N,f^N\rangle\right)\\ &= \sup_{\substack{f^N\in L^2(\nu^N_{\hat\rho})\\ \|f^N\|_{L^2}=1}}\left(\int\gamma\sum_{x\in\mathbb{T}^d_N}\tau_x V_{\epsilon N}(\eta)(f^N)^2(\eta)d\nu^N_{\hat\rho}(\eta)+N^2\langle\mathcal{L}^D_N f^N,f^N\rangle+\langle\mathcal{L}^R_N f^N,f^N\rangle\right)\end{aligned}$$

Estimate the reaction part as follows.

$$\begin{aligned}\langle\mathcal{L}^R_N f^N,f^N\rangle &= \sum_{x\in\mathbb{T}^d_N}\sum_{i=0}^3\int c(x,\eta,i)f^N(\eta)\Big(f^N(\eta^i_x)-f^N(\eta)\Big)d\nu^N_{\hat\rho}(\eta)\\ &= \sum_{x\in\mathbb{T}^d_N}\sum_{i=0}^3\int c(x,\eta,i)f^N(\eta)f^N(\eta^i_x)d\nu^N_{\hat\rho}(\eta)-\sum_{x\in\mathbb{T}^d_N}\sum_{i=0}^3\int c(x,\eta,i)f^N(\eta)^2d\nu^N_{\hat\rho}(\eta)\\ &\leqslant \sum_{x\in\mathbb{T}^d_N}\sum_{i=0}^3\int c(x,\eta,i)\Big(f^N(\eta)^2+\frac14 f^N(\eta^i_x)^2\Big)d\nu^N_{\hat\rho}(\eta)-\sum_{x\in\mathbb{T}^d_N}\sum_{i=0}^3\int c(x,\eta,i)f^N(\eta)^2d\nu^N_{\hat\rho}(\eta)\end{aligned}$$

where we used the inequality $AB\leqslant\dfrac{1}{2a}A^2+\dfrac{a}{2}B^2$ for $A,B,a>0$ with $a=2$. Use formulas of changes of variables given by Lemma 4.B.2 to bound the first sum by the L^2-norm of f^N and the fact that f^N is a density with respect to $\nu^N_{\hat\rho}$ to bound the second integral :

$$\langle\mathcal{L}^R_N f^N,f^N\rangle\leqslant\frac{C(\lambda_1,\lambda_2,r)}{4}\sum_{x\in\mathbb{T}^d_N}\sum_{i=0}^3\sum_{j\neq i}\int f^N(\eta^i_x)^2\eta_j(x)d\nu^N_{\hat\rho}(\eta)$$

$$= \frac{C(\lambda_1,\lambda_2,r)}{4} \sum_{x\in\mathbb{T}_N^d} \sum_{i=0}^{3} \sum_{j\neq i} \int \frac{\rho_j}{\rho_i} f^N(\eta)^2 \eta_i(x) d\nu_{\widehat{\rho}}^N(\eta)$$
$$\leqslant C(\widehat{\rho}) \frac{C(\lambda_1,\lambda_2,r)}{4} N^d$$

Hence,

$$\frac{1}{N^d}\lambda_{\epsilon N}(\gamma) = \sup_{\substack{f^N\in L^2(\nu_{\widehat{\rho}}^N)\\ \|f^N\|_{L^2}=1}} \left(\frac{1}{N^d} \int \gamma \sum_{x\in\mathbb{T}_N^d} \tau_x V_{\epsilon N}(f^N)^2(\eta) d\nu_{\widehat{\rho}}^N(\eta) + N^{2-d}\langle \mathcal{L}_N^D f^N, f^N \rangle \right) + C_0$$

for some positive constant $C_0 = C(\widehat{\rho})C(\lambda_1,\lambda_2,r)/4$. By reversibility of the measure with respect to the generator $\mathcal{L}_N^D$, $\mathbf{D}_N^D(|f^N|) \leqslant \mathbf{D}_N^D(f^N)$ and one can take the supremum over functions f in $L^2(\nu_{\widehat{\rho}}^N)$ such that $\|f\|_{L^2(\nu_{\widehat{\rho}}^N} = 1$ to the supremum over non-negative functions f in $L^2(\nu_{\widehat{\rho}}^N)$ such that $\|\sqrt{f}\|_{L^2(\nu_{\widehat{\rho}}^N} = 1$. Recall $\nu_{\widehat{\rho}}^N$ is reversible with respect to the generator $\mathcal{L}_N^D$ but not $\mathcal{L}_N^R$. Going back to the upper bound of $\lambda_{\epsilon N}(\gamma)$,

$$\frac{1}{N^d}\lambda_{\epsilon N}(\gamma) \leqslant \sup_{\substack{f^N\geqslant 0, f^N\in L^2(\nu_{\widehat{\rho}}^N)\\ \|\sqrt{f^N}\|_{L^2}=1}} \left(\frac{1}{N^d} \int \gamma \sum_{x\in\mathbb{T}_N^d} \tau_x V_{\epsilon N} f^N(\eta) d\nu_{\widehat{\rho}}^N(\eta) + N^{2-d}\langle \mathcal{L}_N^D \sqrt{f^N}, \sqrt{f^N} \rangle \right) + C_0$$
$$\leqslant \sup_{\substack{f^N\geqslant 0, f^N\in L^2(\nu_{\widehat{\rho}}^N)\\ \|\sqrt{f^N}\|_{L^2}=1}} \left(\int \frac{1}{N^d}\gamma \sum_{x\in\mathbb{T}_N^d} \tau_x V_{\epsilon N} f^N(\eta) d\nu_{\widehat{\rho}}^N(\eta) - N^{2-d}\mathbf{D}_N^D(f^N) \right) + C_0,$$

where

$$\mathbf{D}_N^D(f^N) = \sum_{\substack{x,y\in\mathbb{T}_N^d\\ \|x-y\|=1}} \int \left(\sqrt{f^N(\eta^{x,y})} - \sqrt{f^N(\eta)} \right)^2 d\nu_{\widehat{\rho}}^N(\eta)$$

is the Dirichlet form associated to the generator of stirring. Since ϕ is bounded, there exists some positive constant C such that

$$\sum_{x\in\mathbb{T}_N^d} V_{\epsilon N}(\eta) \leqslant CN^d,$$

one can thus restrict the supremum over functions f^N satisfying

$$\mathbf{D}_N^D(f^N) \leqslant CN^{d-2}$$

To get (4.3.17), it remains to show for all positive C,

$$\overline{\lim_{\epsilon\to 0}}\ \overline{\lim_{N\to\infty}} \sup_{f^N\in A_N} \frac{1}{N^d} \int \sum_{x\in\mathbb{T}_N^d} \tau_x V_{\epsilon N}(\eta) f^N(\eta) d\nu_{\widehat{\rho}}^N(\eta) = 0, \tag{4.3.18}$$

where

$$A_N := \left\{ f^N \in L^2(\nu_{\hat\rho}^N) : \ f^N \geqslant 0, \ \|\sqrt{f^N}\|_{L^2} = 1, \mathbf{D}_N^D(f^N) \leqslant CN^{d-2} \right\}$$

This limit will follow from the *blocks estimates*. On one hand, the one block estimate ensures the average of local functions in some large microscopic boxes can be replaced by their mean with respect to the grand-canonical measure parametrized by the particles density in these boxes. While the two blocks estimate ensures the particles density over large microscopic boxes and over small macroscopic boxes is very close. Let us first state the block estimates, we postpone their proofs to the next section.

Lemma 4.3.2 (One block estimate)**.**

$$\varlimsup_{k\to\infty} \varlimsup_{N\to\infty} \sup_{f^N : \mathbf{D}_N^D(f^N) \leqslant CN^{d-2}} \frac{1}{N^d} \int \sum_{x\in\mathbb{T}_N^d} \tau_x \left| \frac{1}{(2k+1)^d} \sum_{\|y\|\leqslant k} \left(\tau_y \phi(\eta) - \widetilde{\phi}(\widehat{\eta}^k(0)) \right) \right| f^N(\eta) d\nu_{\hat\rho}^N(\eta) = 0. \quad (4.3.19)$$

Lemma 4.3.3 (Two blocks estimate)**.** *For $i \in \{0,1,2,3\}$,*

$$\varlimsup_{k\to\infty} \varlimsup_{\epsilon\to 0} \varlimsup_{N\to\infty} \sup_{\|h\|\leqslant \epsilon N} \sup_{f^N : \mathbf{D}_N^D(f^N) \leqslant CN^{d-2}} \frac{1}{N^d} \int \sum_{x\in T_N^d} \left| \eta_i^k(x+h) - \eta_i^{\epsilon N}(x) \right| f^N(\eta) d\nu_{\hat\rho}^N(\eta) = 0. \quad (4.3.20)$$

Let us prove that the limit (4.3.18) is a consequence of these two previous lemmas.

$$\begin{aligned}
&\frac{1}{N^d} \int \sum_{x\in\mathbb{T}_N^d} \tau_x V_{\epsilon N}(\eta) f^N(\eta) d\nu_{\hat\rho}^N(\eta) \\
&= \frac{1}{N^d} \int \sum_{x\in\mathbb{T}_N^d} \tau_x \left| \frac{1}{(2\epsilon N+1)^d} \sum_{\|y\|\leqslant \epsilon N} \tau_y \phi(\eta) - \widetilde{\phi}(\widehat{\eta}^{\epsilon N}(0)) \right| f^N(\eta) d\nu_{\hat\rho}^N(\eta) \\
&\leqslant \frac{1}{N^d} \int \sum_{x\in\mathbb{T}_N^d} \tau_x \left| \frac{1}{(2\epsilon N+1)^d} \sum_{\|y\|\leqslant \epsilon N} \left(\tau_y \phi(\eta) - \frac{1}{(2k+1)^d} \sum_{\|z-y\|\leqslant k} \tau_z \phi(\eta) \right) \right| f^N(\eta) d\nu_{\hat\rho}^N(\eta) \qquad (4.3.21) \\
&\quad + \frac{1}{N^d} \int \sum_{x\in\mathbb{T}_N^d} \tau_x \left| \frac{1}{(2\epsilon N+1)^d} \sum_{\|y\|\leqslant \epsilon N} \left(\frac{1}{(2k+1)^d} \sum_{\|z-y\|\leqslant k} \tau_z \phi(\eta) - \widetilde{\phi}(\widehat{\eta}^k(y)) \right) \right| f^N(\eta) d\nu_{\hat\rho}^N(\eta) \qquad (4.3.22) \\
&\quad + \frac{1}{N^d} \int \sum_{x\in\mathbb{T}_N^d} \tau_x \left| \frac{1}{(2\epsilon N+1)^d} \sum_{\|y\|\leqslant \epsilon N} \widetilde{\phi}(\widehat{\eta}^k(y)) - \widetilde{\phi}(\widehat{\eta}^{\epsilon N}(0)) \right| f^N(\eta) d\nu_{\hat\rho}^N(\eta). \qquad (4.3.23)
\end{aligned}$$

The first expression (4.3.21) of the right-hand side can be decomposed into boxes of size $(2k+1)^d$ so that,

$$\begin{aligned}
&\int \left| \frac{1}{(2\epsilon N+1)^d} \sum_{\|y\|\leqslant \epsilon N} \left(\tau_y \phi(\eta) - \frac{1}{(2k+1)^d} \sum_{\|z-y\|\leqslant k} \tau_z \phi(\eta) \right) \right| f^N(\eta) d\nu^N_{\widehat{\rho}}(\eta) \\
&= \int \left| \frac{1}{(2\epsilon N+1)^d} \sum_{\|y\|\leqslant \epsilon N} \left(\tau_y \phi(\eta) - \frac{1}{(2k+1)^d} \sum_{\|z\|\leqslant k} \tau_{y+z} \phi(\eta) \right) \right| f^N(\eta) d\nu^N_{\widehat{\rho}}(\eta) \\
&= \int \left| \frac{1}{(2k+1)^d} \sum_{\|z\|\leqslant k} \left(\frac{1}{(2\epsilon N+1)^d} \sum_{\|y\|\leqslant \epsilon N} \tau_y \phi(\eta) \right.\right. \\
&\qquad\qquad \left.\left. - \frac{1}{(2\epsilon N+1)^d} \sum_{\|y\|\leqslant \epsilon N} \tau_{y+z} \phi(\eta) \right) \right| f^N(\eta) d\nu^N_{\widehat{\rho}}(\eta) \\
&\leqslant \frac{(2k+1)^d}{(2\epsilon N+1)} \|\phi\|_\infty,
\end{aligned}$$

which tends to zero when N goes to infinity. The second and third expressions (4.3.22) and (4.3.23) tend to zero as well as a consequence of the blocks estimates by translation invariance of $\nu^N_{\widehat{\rho}}$. $\square$

4.3.3 Identification of limit points

Now we show that any limit point of the sequence $(\mathbb{Q}^N_{\mu^N})_{N\geqslant 1}$ is concentrated on trajectories that are weak solutions of the reaction-diffusion system (4.2.14). For this, we come back to the martingale $M^{N,i}_t$ defined in (4.3.1), which satisfies (4.3.9).

We focus on the case $i=1$ since it is the only one for which we need to use the replacement lemma. Define

$$\begin{aligned}
B^1_\epsilon(\widehat{\pi}^N_t) &= \langle \pi^{N,1}_t, G_{1,t}\rangle - \langle \pi^{N,1}_0, G_{1,0}\rangle - \int_0^t \langle \pi^{N,1}_s, \partial_s G_{1,s}\rangle ds - \int_0^t \langle \pi^{N,1}_s, \Delta_N G_{1,s}\rangle ds \\
&- \int_0^t \langle \pi^{N,3}_s, G_{1,s}\rangle ds + \int_0^t (r+1)\langle \pi^{N,1}_s, G_{1,s}\rangle ds \\
&- \int_0^t \frac{1}{N^d} \sum_{x\in\mathbb{T}^d_N} G_{1,s}(x/N) 2d\lambda_1 \langle \pi^{N,1}_s, \iota_\epsilon(\cdot - x/N)\rangle \langle \pi^{N,0}_s, \iota_\epsilon(\cdot - x/N)\rangle ds \\
&- \frac{1}{N^d} \int_0^t \sum_{x\in\mathbb{T}^d_N} G_{1,s}(x/N) 2d\lambda_2 \langle \pi^{N,3}_s, \iota_\epsilon(\cdot - x/N)\rangle \langle \pi^{N,0}_s, \iota_\epsilon(\cdot - x/N)\rangle ds.
\end{aligned}$$

For any $a > 0$, by Doob's inequality,

$$\overline{\lim_{\epsilon\to 0}}\ \overline{\lim_{N\to\infty}}\ \mathbb{P}^N_{\mu^N}\left(\sup_{0\leqslant t\leqslant T}\left|M_t^{N,1}\right| > a\right) = 0$$

To close the equation, replace the local function $\mathcal{L}_N^R\eta_1(x)$ appearing in $M_t^{N,1}$ by a function of the empirical density thanks to the Replacement lemma 4.3.1. Here ϕ is a local function given by

$$\tau_x\phi(\eta) = \left(\lambda_1 \sum_{y:\|y-x\|=1} \eta_1(y) + \lambda_2 \sum_{y:\|y-x\|=1} \eta_3(y)\right)\eta_0(x). \tag{4.3.24}$$

The occupation variables $\eta_i(x)$ are of mean $\eta_i^{\epsilon N}$ under the measure $\nu^N_{\widehat{\eta}^{\epsilon N}}$. Let $\iota_\epsilon = \dfrac{1}{(2\epsilon)^d}\mathbf{1}\{[-\epsilon,\epsilon]^d\}$ be the approximation of the identity and remark that

$$\eta_i^{\epsilon N}(x) = \frac{(2\epsilon N)^d}{(2\epsilon N+1)^d}\langle \pi^{N,i}, \iota_\epsilon(\cdot - x/N)\rangle. \tag{4.3.25}$$

So that, one has by Proposition 4.3.1 and expression (4.3.25) :

$$\overline{\lim_{\epsilon\to 0}}\ \overline{\lim_{N\to\infty}}\ \mathbb{Q}^N_{\mu^N}\Big(\sup_{0\leqslant t\leqslant T}\left|B^1_\epsilon(\widehat{\pi}^N_t)\right| > a\Big) = 0.$$

If $\mathbb{Q}^*_\mu$ is a limit point of the sequence $(\mathbb{Q}^N_{\mu^N})_{N\geqslant 1}$, the mapping $\widehat{\pi} \longmapsto B^1_\epsilon(\widehat{\pi}_T)$ is continuous in Skorohod topology, taking the limit as N goes to infinity,

$$\begin{aligned}\overline{\lim_{\epsilon\to 0}}\ \mathbb{Q}^*_\mu\Bigg(\Bigg|&\langle\pi^1_T, G_{1,T}\rangle - \langle\pi^1_0, G_{1,0}\rangle - \int_0^T \langle\pi^1_s, \partial_s G_{1,s}\rangle ds - \int_0^T\langle\pi^1_s, \Delta G_{1,s}\rangle ds \\ &- \int_0^T\langle\pi^3_s, G_{1,s}\rangle ds + \int_0^T (r+1) < \pi^1_s, G_{1,s}\rangle ds - \int_0^T\int_{\mathbb{T}^d}\Big\{\ G_{1,s}(u)2d\lambda_1\big(\pi^1_s * \iota_\epsilon\big)\big(\pi^0_s * \iota_\epsilon\big)\Big\} ds\ du \\ &- \int_0^T\int_{\mathbb{T}^d}\Big\{\ G_{1,s}(u)2d\lambda_2\big(\pi^3_s * \iota_\epsilon\big)\big(\pi^0_s * \iota_\epsilon\big)\Big\} ds\ du\Bigg| > a\Bigg) = 0.\end{aligned}$$

In virtue of Lemma 4.3.1, all limit points are absolutely continuous with respect to the Lebesgue measure on $\mathbb{T}^d$, that is, if $\widehat{\pi}_s = \widehat{\rho}(s,u)du$, then for all $i \in \{0,1,2,3\}$, $(\pi^i_t * \iota_\epsilon)(u)$ converges in $L^2(\mathbb{T}^d)$ to $\rho_i(t,u)$ as ϵ goes to 0. Hence,

$$\mathbb{Q}^*_\mu\Big(\Big|\langle\pi^1_T, G_{1,T}\rangle - \langle\pi^1_0, G_{1,0}\rangle - \int_0^T\langle\pi^1_s, \partial_s G_{1,s}\rangle ds - \int_0^T\langle\pi^1_s, \Delta G_{1,s}\rangle ds$$

$$- \int_0^T ds \int_{\mathbb{T}^d} du \, \langle \mathfrak{R}_1(\widehat{\pi}_s^1), G_{1,s}(u) \rangle \Big| > a \Big) = 0$$

For $i = 2, 3$, the martingales $M_t^{N,i}$ do not provide local functions of η so that one has immediately the following limits.

$$\mathbb{Q}_\mu^* \Big(\Big| \langle \pi_T^i, G_{i,T}(\cdot) \rangle - \langle \pi_0^i, G_{i,0} \cdot) \rangle - \int_0^T \partial_s \langle \pi_s^i, G_{i,s}(\cdot) \rangle ds - \int_0^T \langle \pi_s^i, \Delta G_{i,s}(\cdot) \rangle ds$$
$$- \int_0^T ds \int_{\mathbb{T}^d} du \, \langle \mathfrak{R}_i(\widehat{\pi}_s^i), G_{i,s}(u) \rangle \Big| > a \Big) = 0$$

Finally, any limit point is concentrated on trajectories $\widehat{\pi}_t(du) = \widehat{\rho}(t,u)du$ which are weak solutions of (4.2.14).

4.3.4 Uniqueness of weak solutions

Following the uniqueness of weak solutions of non-linear parabolic equations done in [42, Appendix 2.4], one has

Proposition 4.3.2. *There exists a unique weak solution to the reaction-diffusion system* (4.2.14) *satisfying (S1) and (S2).*

Proof. For each $z \in \mathbb{Z}^d$, introduce $\psi_z : \mathbb{T}^d \to \mathbb{C}$ defined by

$$\psi_z(u) = \exp\Big((2\pi i)(z.u) \Big) \tag{4.3.26}$$

where $(z.u)$ denotes the usual inner product in $\mathbb{R}^d$. The set $\{\psi_z : z \in \mathbb{Z}^d\}$ forms an orthonormal basis of $L^2(\mathbb{T}^d)$ so that any function $f \in L^2(\mathbb{T}^d)$ can be rewritten as : $f = \sum_{z \in \mathbb{Z}^d} \langle \psi_z, f \rangle \psi_z$, with $\langle ., . \rangle$ standing for the inner product of $L^2(\mathbb{T}^d)$. For any $f, g \in L^2(\mathbb{T}^d)$, one has

$$\int_{\mathbb{T}^d} f(u) g(u) du = \sum_{z \in \mathbb{Z}^d} \langle \psi_z, f \rangle \langle \overline{\psi}_z, g \rangle.$$

Consider now two such solutions of (4.2.14) $\widehat{\rho}^{(1)}$ and $\widehat{\rho}^{(2)}$ starting from an initial profile $\widehat{\gamma}$. Note $\widehat{m}$ the difference $\widehat{\rho}^{(1)} - \widehat{\rho}^{(2)}$ and introduce $R_M^i : [0,T] \to \mathbb{R}$ the function

$$R_M^i(t) = \sum_{z \in \mathbb{Z}^d} \frac{M}{(1 + a|z|^2)(M + a|z|^\alpha)} \langle \psi_z, m_i(t, \cdot) \rangle \langle \overline{\psi}_z, m_i(t, \cdot) \rangle ..$$

Since $\widehat{\rho}^{(j)}$, $j = 1, 2$, satisfies (S1), $R_M^i(t)$ converges as $M \to \infty$ and as $a \to 0$ to

$$R^i(t) := \sum_{z \in \mathbb{Z}^d} \langle \psi_z, m_i(t, \cdot) \rangle \langle \overline{\psi}_z, m_i(t, \cdot) \rangle.$$

which is equal to $\|m_i(t,\cdot)\|^2_{L^2\mathbb{T}^d}$ by (4.3.26). By an integration by parts, note that $\langle \psi_z, \partial_{e_j} f\rangle = -2\pi i z_j \langle \psi_z, f\rangle$, for any function $f \in \mathcal{C}^1(\mathbb{T}^d)$. Now, differentiate $R^i_M(t)$,

$$\begin{aligned}
\partial_t R^i_M(t) &= 2\sum_{z\in\mathbb{Z}^d} \frac{M}{(1+a|z|^2)(M+a|z|^\alpha)} \langle \overline{\psi}_z, m_i(t,\cdot)\rangle\langle \psi_z, \partial_t m_i(t,\cdot)\rangle \\
&= -8\pi^2 \sum_{z\in\mathbb{Z}^d} \frac{M|z|^2}{(1+a|z|^2)(M+a|z|^\alpha)} \langle \overline{\psi}_z, m_i(t,\cdot)\rangle\langle \psi_z, m_i(t,\cdot)\rangle \\
&\quad + \sum_{z\in\mathbb{Z}^d} \frac{M}{(1+a|z|^2)(M+a|z|^\alpha)} \langle \overline{\psi}_z, m_i(t,\cdot)\rangle\langle \psi_z, \mathfrak{R}_i(\widehat{\rho}^{(1)}) - \mathfrak{R}_i(\widehat{\rho}^{(2)})\rangle \\
&\leqslant -8\pi^2 \sum_{z\in\mathbb{Z}^d} \frac{M}{(1+a|z|^2)(M+a|z|^\alpha)} \langle \overline{\psi}_z, m_i(t,\cdot)\rangle\langle \psi_z, m_i(t,\cdot)\rangle \\
&\quad + \sum_{z\in\mathbb{Z}^d} \frac{M}{(1+a|z|^2)(M+a|z|^\alpha)} \langle \overline{\psi}_z, m_i(t,\cdot)\rangle\langle \psi_z, \mathfrak{R}_i(\widehat{\rho}^{(1)}) - \mathfrak{R}_i(\widehat{\rho}^{(2)})\rangle
\end{aligned}$$

where we used that $|z|^2 \geqslant 1$ for all $z \neq 0$. Then, integrating along the time and taking the limit as $M \to \infty$ and $a \to 0$,

$$R^i(t) \leqslant R^i(0) - 8\pi^2 \int_0^t R^i(s)ds + \int_0^t \left|\langle m^i(s,\cdot), \mathfrak{R}_i(\widehat{\rho}^{(1)}) - \mathfrak{R}_i(\widehat{\rho}^{(2)})\rangle\right| ds.$$

Then, notice that $\widehat{\mathfrak{R}}$ is Lipschitz,

$$|\mathfrak{R}_i(\widehat{\rho}^{(1)}) - \mathfrak{R}_i(\widehat{\rho}^{(2)})| \leqslant C(\lambda_1,\lambda_2,r)\sum_{i=1}^3 |m_i|, \text{ for all } i = 1,2,3.$$

Therefore,

$$\sum_{i=1}^3 \|m_i(t,\cdot)\|^2_{L^2(\mathbb{T}^d)} \leqslant \sum_{i=1}^3 \|m_i(0,\cdot)\|^2_{L^2(\mathbb{T}^d)} + 3\big(-8\pi^2 + C(\lambda_1,\lambda_2,r)\big)\sum_{i=1}^3 \int_0^t \|m_i(s,\cdot)\|^2_{L^2(\mathbb{T}^d)} ds$$

and one concludes the proof by Gronwall's inequality. □

4.4 Proof of the replacement lemma

One follows the well-reviewed proofs provided by C. Kipnis and C. Landim [42, Chap. 5], originally introduced by [37].

4.4.1 One block estimate

Proof of Lemma 4.3.2. Note that $V_k(\eta)$ depends only on configurations η through the occupation variables $\{\eta(x), |x| \leqslant k\}$. Therefore, one can project any probability density

f^N on a space of configurations independent of N. Let $\overline{f}^N(\eta) = \frac{1}{N^d} \sum_{x\in\mathbb{T}_N^d} \tau_x f^N(\eta)$. By translation invariance of the measure $\nu_{\widehat{\rho}}^N$,

$$\frac{1}{N^d}\int \sum_{x\in\mathbb{T}_N^d} \tau_x V_k(\eta) f_N(\eta) d\nu_{\widehat{\rho}}^N(\eta) = \int V_k(\eta)\overline{f^N}(\eta) d\nu_{\widehat{\rho}}^N(\eta)$$

For $\Lambda_k := \{x \in \mathbb{T}_N^d, |x| \leqslant k\}$, define $E_k := F^{\Lambda_k}$. Now, denote by $\nu_{\widehat{\rho}}^k$ the product measure $\nu_{\widehat{\rho}}^N$ restricted to E_k and for any probability density f^N, denote by f_k the conditional expectation of f^N with respect to the $\sigma-$algebra $\sigma(\eta(x), x \in \Lambda_k)$, i.e. for all $\eta \in E_k$

$$f_k(\eta) = \frac{1}{\nu_{\widehat{\rho}}^k(\eta)} \int \mathbf{1}\{\eta', \eta'(x) = \eta(x)\ x \in \Lambda_k\} f^N(\eta') d\nu_{\widehat{\rho}}^N(\eta')$$

it is thus enough to show :

$$\overline{\lim_{k\to\infty}}\ \overline{\lim_{N\to\infty}} \sup_{f^N : f^N \leqslant D_N^D(f^N) \leqslant CN^{d-2}} \int V_k(\eta) \overline{f}_k(\eta) d\nu_{\widehat{\rho}}^k(\eta) = 0$$

By convexity of the Dirichlet forms, if D_k^D denotes the Dirichlet form, associated to the stirring process, defined over the set of densities $f_k : \Lambda_k \to \mathbb{R}_+$, then

$$D_k^D(\overline{f}_k) \leqslant C(k) N^{-d} D_N^D(\overline{f}^N) \leqslant C(k) N^{-d} D_N^D(f^N), \tag{4.4.1}$$

so that $D_k^D(\overline{f_k}) \leqslant C'(k) N^{-2}$. Therefore, it remains to show

$$\overline{\lim_{k\to\infty}}\ \overline{\lim_{N\to\infty}} \sup_{f_k : D_k^D(f_k) \leqslant C'(k) N^{-2}} \int V_k(\eta) f_k(\eta) d\nu_{\widehat{\rho}}^k(\eta) = 0$$

By compactness of the level set of D_k^D and lower semi-continuity of Dirichlet forms,

$$\overline{\lim_{N\to\infty}} \sup_{f_k : D_k^D(f_k) \leqslant C'(k) N^{-2}} \int V_k(\eta) f_k(\eta) d\nu_{\widehat{\rho}}^k(\eta) \leqslant \sup_{D_k^D(f_k) = 0} \int V_k(\eta) f_k(\eta) d\nu_{\widehat{\rho}}^k(\eta).$$

Now, it is about to show

$$\overline{\lim_{k\to\infty}} \sup_{f_k : D_k^D(f_k) = 0} \int V_k(\eta) f_k(\eta) d\nu_{\widehat{\rho}}^k(\eta) = 0.$$

A probability density f_k, whose associated Dirichlet form is null, is constant over each hyperplane with a fixed number of type-i particles for all i. The set of measures $\{f_k \nu_{\widehat{\rho}}^k : D_k^D(f_k) = 0\}$ is convex, we can restrict ourselves to its extremal elements which are uniform over the configurations with a fixed number of particles of each type i ($i = 1, 2, 3$). For any vector $\widehat{\ell} = (\ell_1, \ell_2, \ell_3) \in ([0, (2k+1)^d] \cap \mathbb{N}^d)^3$ such that $\ell_1 + \ell_2 +$

$\ell_3 = (2k+1)^d - \ell_0$, denote by $m^k_{\widehat{\ell}}$ the measure $\nu^k_{\widehat{\rho}}$ conditioned on the hyperplane $\{\eta : (2k+1)^d \widehat{\eta}^k(0) = \widehat{\ell}\}$,

$$m^k_{\widehat{\ell}}(\cdot) = \nu^k_{\widehat{\rho}}\Big(\ \cdot \ \big|(2k+1)^d \widehat{\eta}^k(0) = \widehat{\ell}\Big) \tag{4.4.2}$$

Note this measure does not depend on $\widehat{\rho}$. It remains to show

$$\varlimsup_{k\to\infty} \sup_{\widehat{\ell}} \int \Big| \frac{1}{(2k+1)^d} \sum_{\|y\|\leqslant k} \tau_y \phi(\eta) - \widetilde{\phi}\Big(\frac{\widehat{\ell}}{(2k+1)^d}\Big)\Big| dm^k_{\widehat{\ell}}(\eta) = 0. \tag{4.4.3}$$

Now fix a positive integer p increasing to infinity after k and decompose the set Λ_k in cubes of length $(2p+1)$. Consider the set $A = \{(2p+1)x, x \in \mathbb{Z}^d\} \bigcap \Lambda_{k-p}$ and list its elements by $A = \{x_1, ..., x_q\}$ such that $\|x_\ell\| \leqslant \|x_j\|$ for $\ell \leqslant j$. Let $B_\ell = x_\ell + \Lambda_p$ if $1 \leqslant \ell \leqslant q$. Note that $B_\ell \bigcap B_j = \varnothing$ if $\ell \neq j$ and $\bigcup_{\ell=1}^{q} B_\ell \subset \Lambda_k$. Define $B_0 = \Lambda_k \backslash \bigcup_{\ell=1}^{q} B_\ell$ so that $|B_0| \leqslant Cpk^{d-1}$ by construction, for some positive constant C. This way, the integral (4.4.3) is bounded by

$$\sum_{i=1}^{q} \frac{|B_\ell|}{|\Lambda_k|} \int \Big| \frac{1}{|B_\ell|} \sum_{y\in B_\ell} \tau_y \phi(\eta) - \widetilde{\phi}\Big(\frac{\widehat{\ell}}{(2k+1)^d}\Big)\Big| dm^k_{\widehat{\ell}}(\eta)$$

But $|B_0| \leqslant Cpk^{d-1}$ and occupation variables $\eta_i(x)$ have mean $\ell_i/(2k+1)^d$ under $m^k_{\widehat{\ell}}$,

$$\sum_{\ell=1}^{q} \frac{|\Lambda_p|}{|\Lambda_k|} \int \Big| \frac{1}{|\Lambda_p|} \sum_{y\in B_\ell} \tau_y \phi(\eta) - \widetilde{\phi}\Big(\frac{\widehat{\ell}}{(2k+1)^d}\Big)\Big| dm^k_{\widehat{\ell}}(\eta) + O(p/k)$$

Moreover, the distribution of the occupation variables $\{(\xi, \omega)(y), y \in B_\ell\}$ do not depend on ℓ, this sum is hence equal to

$$\int \Big| \frac{1}{(2p+1)^d} \sum_{\|y\|\leqslant p} \tau_y \phi(\eta) - \widetilde{\phi}\Big(\frac{\widehat{\ell}}{(2k+1)^d}\Big)\Big| dm^k_{\widehat{\ell}}(\eta) + O(p/k)$$

By the equivalence of ensembles (see next Lemma 4.4.1), letting k go to infinity and $\widehat{\ell}/(2k+1)^d$ tend to $\widehat{\rho}$, this integral converges to

$$\int \Big| \frac{1}{(2p+1)^d} \sum_{\|y\|\leqslant p} \tau_y \phi(\eta) - \widetilde{\phi}\big(\widehat{\rho}\big)\Big| d\nu_{\widehat{\rho}}(\eta) \tag{4.4.4}$$

As p goes to infinity, since $\nu_{\widehat{\rho}}$ is product, by the law of large numbers this integral converges uniformly to 0 on every compact subset of $\mathbb{R}_+$. $\square$

4.4.2 Equivalence of ensembles

To prove the closeness between the grand-canonical and the canonical measures, we derive the so-called equivalence of ensembles.

Lemma 4.4.1 (Equivalence of ensembles). *For every bounded function $f : \{0,1,2,3\}^{\mathbb{T}^d} \to \mathbb{R}$,*

$$\lim_{k\to\infty} \sup_{\hat{\ell}} \left| m^k_{\hat{\ell}}(f) - \nu^k_{\hat{\ell}/(2k+1)^d}(f) \right| = 0$$

Proof of the equivalence of ensembles. For any $m \in \mathbb{N}$, let $(x_1, ..., x_m) \in (\Lambda_k)^m$ and let $m_i = \sum_{j=1}^m \eta_i(x_j)$. Denote by I_i the set of sites that are in state $i \in \{1,2,3\}$, i.e. $I_i = \{x_j, j = 1, ..., m : \eta_i(x_j) = 1\}$, so that $|I_i| = m_i$.

Consider $\ell_0 = (2k+1)^d - \ell_1 - \ell_2 - \ell_3$ and $m_0 = m - m_1 - m_2 - m_3$. First, compute

$$\begin{aligned}
\nu^k_{\hat{\rho}}\Bigg(& \eta_1(x_j) = 1, x_j \in I_1 \ ; \ \eta_2(x_j) = 1, x_j \in I_2 \ ; \ \eta_3(x_j) = 1, x_j \in I_3 \ ; \\
& \sum_{\Lambda_k \setminus \bigcup_{i=1}^3 I_i} \eta_1(x) = \ell_1 - m_1 \ ; \ \sum_{\Lambda_k \setminus \bigcup_{i=1}^3 I_i} \eta_2(x) = \ell_2 - m_2 \ ; \ \sum_{\Lambda_k \setminus \bigcup_{i=1}^3 I_i} \eta_3(x) = \ell_3 - m_3 \Bigg) \\
&= \frac{\big((2k+1)^d - m\big)!}{(\ell_0 - m_0)!(\ell_1 - m_1)!(\ell_2 - m_2)!(\ell_3 - m_3)!} (\varrho_0)^{\ell_0} (\varrho_1)^{\ell_1} (\varrho_2)^{\ell_2} (\varrho_3)^{\ell_3}
\end{aligned}$$

by the expression of the measure $\nu^N_{\hat{\rho}}$ given in (4.2.8) and

$$\begin{aligned}
\nu^k_{\hat{\rho}}\Bigg(\sum_{x\in\Lambda_k} \eta_1(x) = \ell_1 \ ; \ \sum_{x\in\Lambda_k} \eta_2(x) = \ell_2 \ ; \ \sum_{x\in\Lambda_k} \eta_3(x) = \ell_3 \Bigg) \\
= \frac{(2k+1)^d}{\ell_0!\ell_1!\ell_2!\ell_3!} (\varrho_0)^{\ell_0} (\varrho_1)^{\ell_1} (\varrho_2)^{\ell_2} (\varrho_3)^{\ell_3}
\end{aligned}$$

Consequently, the canonical measure is given by

$$\begin{aligned}
m^k_{\hat{\ell}} & \Big(\eta_1(x_j) = 1, x_j \in I_1 \ ; \ \eta_2(x_j) = 1, x_j \in I_2 \ ; \ \eta_3(x_j) = 1, x_j \in I_3 \Big) \\
&= \frac{\big((2k+1)^d - m\big)!}{(2k+1)^d!} \frac{\ell_0!}{(\ell_0 - m_0)!} \frac{\ell_1!}{(\ell_1 - m_1)!} \frac{\ell_2!}{(\ell_2 - m_2)!} \frac{\ell_3!}{(\ell_3 - m_3)!}
\end{aligned}$$

while the grand-canonical measure is defined by

$$\begin{aligned}
\nu^k_{\hat{\ell}/(2k+1)^d} & (\eta_1(x_j) = 1, x_j \in I_1 \ ; \ \eta_2(x_j) = 1, x_j \in I_2 \ ; \ \eta_3(x_j) = 1, x_j \in I_3) \\
&= \left(\frac{\ell_0}{(2k+1)^d}\right)^{m_0} \left(\frac{\ell_1}{(2k+1)^d}\right)^{m_1} \left(\frac{\ell_2}{(2k+1)^d}\right)^{m_2} \left(\frac{\ell_3}{(2k+1)^d}\right)^{m_3}
\end{aligned}$$

Recall that $\Upsilon^k_{\widehat{\ell}} = \left| m^k_{\widehat{\ell}} - \nu^k_{\widehat{\ell}/(2k+1)^d} \right|$,

$$\begin{aligned}
&\Upsilon^k_{\widehat{\ell}}\Big(\eta_1(x_j) = 1, x_j \in I_1 \ ; \ \eta_2(x_j) = 1, x_j \in I_2 \ ; \ \eta_3(x_j) = 1, x_j \in I_3\Big) \\
&= \left(\frac{\prod_{i=0}^{3} (\ell_i)^{m_i}}{((2k+1)^d)^m} \right) \left(\left(\frac{\prod_{i=0}^{3} \frac{\ell_i}{\ell_i} \frac{\ell_i - 1}{\ell_i} \dots \frac{\ell_i - m_i + 1}{\ell_i}}{\frac{(2k+1)^d}{(2k+1)^d} \frac{(2k+1)^d - 1}{(2k+1)^d} \dots \frac{(2k+1)^d - m + 1}{(2k+1)^d}} \right) - 1 \right) \\
&= \left(\frac{\prod_{i=0}^{3} (\ell_i)^{m_i}}{((2k+1)^d)^m} \right) \left(\left(\frac{\prod_{i=0}^{3} \Big(1 - \frac{1}{\ell_i}\Big) \dots \Big(1 - \frac{m_i - 1}{\ell_i}\Big)}{\Big(1 - \frac{1}{(2k+1)^d}\Big) \dots \Big(1 - \frac{m-1}{(2k+1)^d}\Big)} \right) - 1 \right)
\end{aligned}$$

Taking now the maximum over $\widehat{\ell} \in (0, .., (2k+1)^d)^3$,

$$\begin{aligned}
\max_{\widehat{\ell}} \Upsilon^k_{\widehat{\ell}}&\Big(\eta_1(x_j) = 1, x_j \in I_1 \ ; \ \eta_2(x_j) = 1, x_j \in I_2 \ ; \ \eta_3(x_j) = 1, x_j \in I_3\Big) \\
&\leqslant \left(\frac{1}{\Big(1 - \frac{1}{(2k+1)^d}\Big) \dots \Big(1 - \frac{m-1}{(2k+1)^d}\Big)} \right) - 1
\end{aligned}$$

which tends to zero as $k \to \infty$. □

4.4.3 Two blocks estimate

Proof of Lemma 4.3.3. Begin by replacing the average over a small macroscopic box of size $(2\epsilon N + 1)^d$ by the average over large microscopic boxes of size $(2k+1)^d$, that is, for N large enough, one has

$$\begin{aligned}
&\Big| \eta_i^k(h) - \eta_i^{\epsilon N}(0) \Big| \\
&\leqslant \Big| \frac{1}{(2k+1)^d} \sum_{\|y-h\| \leqslant k} \eta_i(y) - \frac{1}{(2\epsilon N+1)^d} \sum_{\|y\| \leqslant \epsilon N} \frac{1}{(2k+1)^d} \sum_{\|z-y\| \leqslant k} \eta_i(z) \Big| \\
&\quad + \Big| \frac{1}{(2\epsilon N+1)^d} \sum_{\|y\| \leqslant \epsilon N} \frac{1}{(2k+1)^d} \sum_{\|z-y\| \leqslant k} \eta_i(z) - \frac{1}{(2\epsilon N+1)^d} \sum_{\|y\| \leqslant \epsilon N} \eta_i(y) \Big| \\
&\leqslant \sup_{2k < \|h\| \leqslant 2\epsilon N} \Big| \eta_i^k(h) - \eta_i^k(0) \Big| + \frac{(2k+1)^d}{2\epsilon N + 1}
\end{aligned}$$

It is thus enough to show :

$$\begin{aligned}
&\overline{\lim_{k\to\infty}}\ \overline{\lim_{\epsilon\to 0}}\ \overline{\lim_{N\to\infty}} \sup_{f^N : D_N^D(f^N) \leqslant C N^{D-2}} \ \sup_{2k+1 \leqslant \|h\| \leqslant 2\epsilon N} \\
&\qquad \frac{1}{N^d} \int \sum_{x \in \mathbb{T}_N^d} \tau_x \Big| \eta_i^k(0) - \eta_i^k(h) \Big| f^N(\eta) d\nu^N_{\widehat{\rho}}(\eta) = 0 \quad (4.4.5)
\end{aligned}$$

By translation invariance of the measure, one can rewrite the integral as

$$\int \left|\eta_i^k(0) - \eta_i^k(h)\right| \bar{f}^N(\eta) d\nu_{\hat{\rho}}^N(\eta)$$

where $\eta_i^k(0)$ and $\eta_i^k(h)$ depend only on configurations (η) over the set of occupation variables $\{\eta(x), x \in \Lambda_{h,k}\}$, with $\Lambda_{h,k} := \{-k, ..., k\}^d \cup (h + \{-k, ..., k\}^d)$.

Denote by $\nu_{\hat{\rho}}^{2k}$ the product measure $\nu_{\hat{\rho}}^N$ restricted to $E_k \times E_k$ and for any density f^N, denote by $f_{h,k}$ the conditional expectation of f^N with respect to the sigma-algebra $\sigma(\eta(x), x \in \Lambda_{h,k})$. Let ζ and χ be two copies of η defined on E_k, it is enough to prove

$$\overline{\lim_{k\to\infty}}\ \overline{\lim_{\epsilon\to 0}}\ \overline{\lim_{N\to\infty}} \sup_{f^N: D_N^D(f^N) \leqslant CN^{D-2}} \sup_{2k+1 \leqslant \|h\| \leqslant 2\epsilon N} \int \left|\zeta_i^k(0) - \chi_i^k(0)\right| \overline{f_{h,k}}(\zeta, \chi) d\nu_{\hat{\rho}}^{2k}(\zeta, \chi) = 0 \quad (4.4.6)$$

Let g be a function on $E_k \times E_k$, define the following Dirichlet forms corresponding to exchanges within two separate boxes and to exchanges between those two boxes, for two neighbouring sites $x, y \in \Lambda_k$

$$\begin{aligned}
\mathbf{D}_{x,y}^{1,k}(g) &= \int \left(\sqrt{g(\zeta^{x,y}, \chi)} - \sqrt{g(\zeta, \chi)}\right)^2 d\nu_{\hat{\rho}}^{2k}(\zeta, \chi) \\
\mathbf{D}_{x,y}^{2,k}(g) &= \int \left(\sqrt{g(\zeta, \chi^{x,y})} - \sqrt{g(\zeta, \chi)}\right)^2 d\nu_{\hat{\rho}}^{2k}(\zeta, \chi) \\
\Delta_k(g) &= \int \left(\sqrt{g^k((\zeta, \chi)^0)} - \sqrt{g^k(\zeta, \chi)}\right)^2 d\nu_{\hat{\rho}}^{2k}(\zeta, \chi)
\end{aligned}$$

where $(\zeta, \chi)^0$ is obtained from (ζ, χ) by switching the values of $\zeta(0)$ and $\chi(0)$. Define

$$\mathfrak{D}_k(g) = \mathbf{D}_{x,y}^{1,k}(g) + \mathbf{D}_{x,y}^{1,k}(g) + \Delta_k(g) \quad (4.4.7)$$

As for the one block estimate, one has the following upper bounds. For all $x, y \in \Lambda_k$ such that $\|x - y\| = 1$,

$$\mathbf{D}_{x,y}^{1,k}(f_{h,k}) \leqslant \mathbf{D}_{x,y}^{D}(f^N), \quad \text{and} \quad \mathbf{D}_{x,y}^{2,k}(f_{h,k}) \leqslant \mathbf{D}_{h+x,h+y}^{D}(f^N)$$

As in (4.4.1), summing over each pair $x, y \in \Lambda_k$ such that $\|x - y\| = 1$:

$$\sum_{x,y \in \Lambda_k : \|x-y\|=1} \mathbf{D}_{x,y}^{1,k}(\overline{f}_{h,k}) + \sum_{x,y \in \Lambda_k : \|x-y\|=1} \mathbf{D}_{x,y}^{1,k}(\overline{f}_{h,k}) \leqslant 2C(k)N^{-d} D_N^D(f^N) \leqslant C(k)N^{-2},$$

for any probability density whose Dirichlet form is bounded by CN^{d-2}. For the last one,

$$\Delta_k(\overline{f}_{h,k}) \leqslant \int \left(\sqrt{\overline{f}^N(\eta^{0,h})} - \sqrt{\overline{f}^N(\eta)}\right)^2 d\nu_{\hat{\rho}}^N(\eta). \quad (4.4.8)$$

To switch the occupations variables of $\zeta(0)$ and $\chi(0)$, define a path from the origin to h by a sequence of sites $x_0, ..., x_{\|h\|_1}$ such that $x_0 = 0$, $x_{\|h\|_1} = h$ and for each $0 \leqslant k \leqslant \|h\|_1 - 1$, $\|x_{k+1} - x_k\|_1 = 1$, so that we have a telescopic summation

$$\sqrt{\overline{f}^N(\eta^{0,h})} - \sqrt{\overline{f}^N(\eta)} = \sum_{k=0}^{\|h\|_1-1} \left(\sqrt{\overline{f}^N(\eta^{x_k,x_{k+1}})} - \sqrt{\overline{f}^N(\eta)}\right).$$

By Cauchy-Schwarz inequality, from (4.4.8)

$$\Delta_k(\overline{f}_{h,k}) \leqslant \|h\|_1 \sum_{k=0}^{\|h\|_1-1} \int \left(\sqrt{\overline{f}^N(\eta^{x_k,x_{k+1}})} - \sqrt{\overline{f}^N(\eta)}\right)^2 d\nu_{\hat{\rho}}^N(\eta)$$

which is equal to $\|h\|_1 \sum_{k=0}^{\|h\|_1-1} \mathbf{D}^D_{x_k,x_{k+1}}(\overline{f}^N)$. From (4.4.1), $\mathbf{D}^D_{x_k,x_{k+1}}(\overline{f}^N) \leqslant N^{-d}\mathbf{D}^D_N(\overline{f}^N)$. Moreover, $\|h\|_1 \leqslant 2\epsilon N$, hence

$$\Delta_k(\overline{f}_{h,k}) \leqslant \|h\|_1^2 N^{-d}\mathbf{D}^D_N(f^N) \leqslant C_1(2\epsilon)^2.$$

To conclude the proof, it is thus enough to show that

$$\overline{\lim_{k\to\infty}}\ \overline{\lim_{\epsilon\to 0}} \sup_{\mathfrak{D}_k(f)\leqslant C(k)\epsilon^2} \int \left|\zeta^k(0) - \chi^k(0)\right| f(\zeta,\chi) d\nu_{\hat{\rho}}^{2k}(\zeta,\chi) = 0 \tag{4.4.9}$$

We conclude as for the 1-block estimate : we first let ϵ go to zero, then if f satisfies $\mathfrak{D}_k(f) = 0$, it is constant on hyperplanes having a fixed total number of particles of each type i on $\Lambda_k \cup \left(h + \Lambda_k\right)$. The result is a consequence of the equivalence of ensembles. □

4.A Construction of an auxiliary process

The reference measure $\nu_{\hat{\rho}}^N$ defined in (4.2.6) is only reversible with respect to the generator of stirring $\mathcal{L}_N^D$. Assuming the occupation variables are unbounded, we would not be able to use the bound of the proof of the replacement lemma 4.3.1, a way to avoid this issue is to build an auxiliary reaction process whose generator is invariant (or reversible if the dynamics makes it possible, but this is not our case) with respect to the reference measure. We follow arguments presented by M. Mourragui [63], for births, deaths and jump processes.

Construction of the generator. It is about to construct a convenient transition function $\widetilde{c}$ for which the measure $\nu_{\hat{\rho}}^N$ is invariant with respect to an auxiliary Markov process with generator $\widetilde{\mathcal{L}}_N^R$, that is for any function f on E_N

$$\int \widetilde{\mathcal{L}}_N^R f(\eta) d\nu_{\hat{\rho}}^N(\eta) = 0. \tag{4.A.1}$$

Let $\widetilde{r}(x,\eta) = r_0\mathbf{1}\{\eta(x)=0\} + r_1\mathbf{1}\{\eta(x)=1\}$, δ_1, δ_2 and α be parameters associated to the generator $\widetilde{\mathcal{L}}_N^R$ to determine. By a change of variables [see Lemma 4.B.2],

$$
\begin{aligned}
&\int \widetilde{\mathcal{L}}_N^R f(\eta) d\nu_{\widehat{\rho}}^N(\eta) \\
&= \int \sum_{x\in T_N^d} (\alpha[f(\eta_x^1)-f(\eta)] + \widetilde{r}(x,\eta)[f(\eta_x^2)-f(\eta)])\mathbf{1}\{\eta(x)=0\} d\nu_{\widehat{\rho}}^N(\eta) \\
&\quad + \int \sum_{x\in T_N^d} (\delta_1[f(\eta_x^0)-f(\eta)] + \widetilde{r}(x,\eta)[f(\eta_x^3)-f(\eta)])\mathbf{1}\{\eta(x)=1\} d\nu_{\widehat{\rho}}^N(\eta) \\
&\quad + \int \sum_{x\in T_N^d} \delta_2[f(\eta_x^0)-f(\eta)]\mathbf{1}\{\eta(x)=2\} d\nu_{\widehat{\rho}}^N(\eta) \\
&\quad + \int \sum_{x\in T_N^d} (\delta_2[f(\eta_x^1)-f(\eta)] + \delta_1[f(\eta_x^2)-f(\eta)])\mathbf{1}\{\eta(x)=3\} d\nu_{\widehat{\rho}}^N(\eta) \\
&= \int \sum_{x\in T_N^d} f(\eta)\left[\mathbf{1}\{\eta(x)=0\}\left(\delta_1\frac{\rho_1}{\rho_0} + \delta_2\frac{\rho_2}{\rho_0} - \alpha - r_0\right)\right. \\
&\quad + \mathbf{1}\{\eta(x)=1\}\left(\alpha\frac{\rho_0}{\rho_1} + \delta_2\frac{\rho_3}{\rho_1} - \delta_1 - r_1\right) + \mathbf{1}\{\eta(x)=2\}\left(r_0\frac{\rho_0}{\rho_2} + \delta_1\frac{\rho_3}{\rho_2} - \delta_2\right) \\
&\quad \left. + \mathbf{1}\{\eta(x)=3\}\left(r_1\frac{\rho_1}{\rho_3} - \delta_2 - \delta_1\right)\right] d\nu_{\widehat{\rho}}(\eta).
\end{aligned}
$$

A sufficient condition for this integral to be null is that each term between brackets vanishes. Therefore, posing $\delta_1 = \delta_2 = 1$, the measure $\nu_{\widehat{\rho}}^N$ is invariant with respect to $\widetilde{\mathcal{L}}_N^R$ as soon as

$$
\widetilde{r}(x,\eta) = \frac{\rho_2-\rho_3}{\rho_0}\mathbf{1}\{\eta(x)=0\} + 2\frac{\rho_3}{\rho_1}\mathbf{1}\{\eta(x)=1\} \tag{4.A.2}
$$

and

$$
\alpha = \frac{\rho_1+\rho_3}{\rho_0}. \tag{4.A.3}
$$

If $\rho_2 > \rho_3$, the rate $\widetilde{r}(x,\eta)$ is well defined. Subsequently, fix such a profile $\widehat{\rho}$ to define the reference measure $\nu_{\widehat{\rho}}^N$. Fix the dynamics with parameters $\widetilde{r}(x,\eta)$ and α satisfying (4.A.2)-(4.A.3), i.e.

$$
\begin{array}{ll}
0 \to 1 \text{ at rate } \alpha & 1 \to 0 \text{ at rate } 1 \\
0 \to 2 \text{ at rate } r_0 & 2 \to 0 \text{ at rate } 1 \\
1 \to 3 \text{ at rate } r_1 & 3 \to 1 \text{ at rate } 1 \\
 & 3 \to 2 \text{ at rate } 1
\end{array} \tag{4.A.4}
$$

One can thus construct uniquely a Markov process with generator

$$
\widetilde{\mathfrak{L}}_N = N^2\mathcal{L}_N^D + \widetilde{\mathcal{L}}_N^R,
$$

that admits $\nu_{\hat{\rho}}^N$ defined in (4.2.6) as unique invariant measure, this is the so-called *auxiliary process*.

Denote by $\widetilde{\mathbb{P}}^N_{\nu^N_{\hat{\rho}}}$ the probability measure of the auxiliary process starting from the initial measure $\nu_{\hat{\rho}}^N$ and by $\widetilde{\mathbb{E}}^N_{\nu^N_{\hat{\rho}}}$ the corresponding expectation. In view of the dynamics of the reaction part, there is no way to build a generator that is reversible with respect to the reference measure, this would though be possible for the symmetric CP-DRE, as we will discuss in the next chapter.

Entropy of $\mathbb{P}^N_{\mu^N}$ with respect to $\widetilde{\mathbb{P}}^N_{\nu^N_{\hat{\rho}}}$. Start by defining $\mathbf{H}(\mathbb{P}^N_{\mu^N}|\widetilde{\mathbb{P}}^N_{\nu^N_{\hat{\rho}}})$ the entropy of $\mathbb{P}^N_{\mu}$ with respect to $\widetilde{\mathbb{P}}^N_{\nu^N_{\hat{\rho}}}$ as the positive convex function given by

$$\mathbf{H}(\mathbb{P}^N_{\mu^N}|\widetilde{\mathbb{P}}^N_{\nu^N_{\hat{\rho}}}) = \int \log \frac{\mathbb{P}^N_{\mu^N}}{\widetilde{\mathbb{P}}^N_{\nu^N_{\hat{\rho}}}} d\mathbb{P}^N_{\mu^N}(\eta). \tag{4.A.5}$$

Controlling the relative entropy of $\mathbb{P}^N_{\mu^N}$ with respect to $\widetilde{\mathbb{P}}^N_{\nu^N_{\hat{\rho}}}$ allows us to deduce properties of the reaction-diffusion process from results settled for the auxiliary process via the entropy inequality. This inequality is given for any bounded continuous function U by

$$\int U(\cdot)d\mathbb{P}^N_{\mu^N}(\cdot) \leqslant \log \int \exp(U(\cdot))d\widetilde{\mathbb{P}}^N_{\nu^N_{\hat{\rho}}}(\cdot) + \mathbf{H}(\mathbb{P}^N_{\mu^N}|\widetilde{\mathbb{P}}^N_{\nu^N_{\hat{\rho}}}) \tag{4.A.6}$$

Since the occupation variables are bounded, by convexity of the entropy (see [42, Appendix I.8]),

$$\begin{aligned} \mathbf{H}(\mu^N|\nu^N_{\hat{\rho}}) &\leqslant \sum_{\eta\in E_N} \mu^N(\eta)\mathbf{H}(\delta_\eta|\nu^N_{\hat{\rho}}) = \sum_{\eta\in E_N} \mu^N(\eta)\log\left(\frac{1}{\nu^N_{\hat{\rho}}(\eta)}\right) \\ &\leqslant \sum_{\eta\in E_N} \mu^N(\eta)\log\left(\frac{1}{\inf_i \rho_i}\right)^{N^d} = C_0 N^d, \end{aligned} \tag{4.A.7}$$

for some positive constant C_0.

To study the entropy of $\mathbb{P}^N_{\mu^N}$ with respect to $\widetilde{\mathbb{P}}^N_{\nu^N_{\hat{\rho}}}$, begin by computing the associated Radon-Nikodym density. For this, introduce the following jump processes corresponding to each transition of the reaction part :

- $D_t^{x,i}$: number of deaths of type-i particles on site x up to time t, for $i = 1, 2$.
- B_t^x : number of births of type-1 particles on site x up to time t.
- $I_t^{x,j}$: number of arrivals of type-2 particles on site x in state j up to time t, for $j = 0, 1$.

Then, $\widetilde{D}_t^{x,i} = D_t^{x,i} - \int_0^t \mathbf{1}\{\eta_s(x) = i\}ds$, $\widetilde{I}_t^{x,j} = I_t^{x,j} - \int_0^t \widetilde{r}_s(x)ds$ and $\widetilde{B}_t^x = B_t^x - \alpha\int_0^t \mathbf{1}\{\eta_s(x) = 0\}ds$ are $\widetilde{\mathbb{P}}^N$-martingales.

Furthermore, $\widetilde{D}_t^{x,i}$, $\hat{I}_t^{x,j} = I_t^{x,j} - r\int_0^t \mathbf{1}\{\eta_s(x) = j\}ds$ and $\widehat{B}_t^x = B_t^x - \int_0^t \Big(\lambda_1 n_1(x,\eta_s) + \lambda_2 n_3(x,\eta_s)\Big)\mathbf{1}\{\eta_s(x) = 0\}ds$ are $\mathbb{P}^N$-martingales. Remark that, since $n_i(x,\eta) \leqslant 2d$ and $\lambda_2 < \lambda_1$, one has $\lambda_1 n_1(x,\eta) + \lambda_2 n_3(x,\eta) \leqslant 2d\lambda_1$, for all $x \in \mathbb{T}_N^d$. Rates $\widetilde{r}(x)$ and α were defined in (4.A.2)-(4.A.3).

Starting from a common initial measure, one obtains the density via the Girsanov formula for jump processes [42, Proposition A1.2.6]. Since $D_t^{x,i}$ have same jump rate, both are $\mathbb{P}^N$- and $\widetilde{\mathbb{P}}^N$- martingales, so that they vanish in the computation of the density while on the other hand,

$$\begin{aligned}
\frac{d\mathbb{P}^N_{\nu^N_{\hat\rho}}}{d\widetilde{\mathbb{P}}^N_{\nu^N_{\hat\rho}}}(\eta_\cdot) = \exp\Bigg\{ \sum_{x\in\mathbb{T}_N^d} \Bigg(\Bigg(\int_0^t \log\frac{r}{r_0} dI_s^{x,0} - \int_0^t \big(r - r_0\big)\mathbf{1}\{\eta_s(x) = 0\}ds + \\
\int_0^t \log\frac{r}{r_1} dI_s^{x,1} - \int_0^t \big(r - r_1\big)\mathbf{1}\{\eta_s(x) = 1\}ds \\
+ \int_0^t \log\left(\frac{\lambda_1 n_1(x,\eta_s) + \lambda_2 n_3(x,\eta_s)}{\alpha}\right) dB_s^x \\
- \int_0^t \Bigg(\lambda_1 n_1(x,\eta_s) + \lambda_2 n_3(x,\eta_s) - \alpha\Bigg)\mathbf{1}\{\eta_s(x) = 1\}ds\Bigg)\Bigg\},
\end{aligned} \tag{4.A.8}$$

where the stochastic integral of a bounded continuous function f with respect to a jump process $(I_t)_{t\geqslant 0}$ is defined by

$$\int_0^t f(\eta_s)dI_s = \sum_{s\leqslant t} f(\eta_{s-})(I_s - I_{s-})$$

Proposition 4.A.1. *There exists a positive constant C such that*

$$\mathbf{H}(\mathbb{P}^N_{\mu^N}|\widetilde{\mathbb{P}}^N_{\nu^N_{\hat\rho}}) \leqslant CN^d.$$

Proof. By definition of the entropy

$$\mathbf{H}(\mathbb{P}^N_{\mu^N}|\widetilde{\mathbb{P}}^N_{\nu^N_{\hat\rho}}) \quad = \quad \int \log\left(\frac{d\mu^N}{d\nu^N_{\hat\rho}}(\eta_0)\frac{d\mathbb{P}^N_{\nu^N_{\hat\rho}}}{d\widetilde{\mathbb{P}}^N_{\nu^N_{\hat\rho}}}(\eta_\cdot)\right)d\mathbb{P}^N_{\mu^N}(\eta_\cdot)$$

$$= \mathbf{H}(\mu^N|\nu_{\widehat{\rho}}^N) + \int \log\left(\frac{d\mathbb{P}^N_{\nu_{\widehat{\rho}}^N}}{d\widetilde{\mathbb{P}}^N_{\nu_{\widehat{\rho}}^N}}(\eta_\cdot)\right) d\mathbb{P}^N_{\mu^N}(\eta_\cdot)$$

Using (4.A.7), the result comes from (4.A.8) since the involved rates are bounded :

$$\begin{aligned}
&\int \log\left(\frac{d\mathbb{P}^N_{\nu_{\widehat{\rho}}^N}}{d\widetilde{\mathbb{P}}^N_{\nu_{\widehat{\rho}}^N}}(\eta_\cdot)\right) d\mathbb{P}^N_{\mu^N}(\eta_\cdot)\\
&\leqslant C(\lambda_1,\lambda_2,r,r_0,r_1,\alpha)\sum_{x\in\mathbb{T}_N^d}\left(\mathbb{E}^N_{\mu^N}\left(B_t^x\right)+\mathbb{E}^N_{\mu^N}\left(I_t^{x,0}\right)+\mathbb{E}^N_{\mu^N}\left(I_t^{x,1}\right)\right)\\
&\leqslant C'N^d
\end{aligned}$$

□

First, prove this limit for the auxiliary process with infinitesimal generator $\widetilde{\mathfrak{L}}_N$. Next, one concludes for the reaction-diffusion process using the entropy inequality given by (4.A.6). It is now about to prove the following.

Replacement lemma In a more suitable way, one can now prove the replacement lemma 4.3.1 for the process of generator $\widetilde{\mathfrak{L}_N}$. After what we deduce the result for the reaction-diffusion process of generator $\mathfrak{L}_N$ by inequality entropy using 4.A.7 and 4.A.1.

Proposition 4.A.2. *For all $a > 0$,*

$$\overline{\lim_{\epsilon\to 0}}\ \overline{\lim_{N\to\infty}}\ \frac{1}{N^d}\log\widetilde{\mathbb{P}}^N_{\nu_{\widehat{\rho}}^N}\left(\frac{1}{N^d}\int_0^T\sum_{x\in\mathbb{T}_N^d}\tau_x V_{\epsilon N}(\eta_t)dt \geqslant a\right) = -\infty \tag{4.A.9}$$

Proof. The proof is very similar to the proof of Proposition 4.3.1 with the exception of estimating the term $\langle\widetilde{\mathcal{L}}_N^R\sqrt{f^N},\sqrt{f^N}\rangle$. This is done as following.

$$\begin{aligned}
\langle\widetilde{\mathcal{L}}_N^R\sqrt{f^N},\sqrt{f^N}\rangle &= \sum_{x\in\mathbb{T}_N^d}\sum_{i=0}^{3}\int\widetilde{c}(x,\eta,i)\sqrt{f^N(\eta)}\Big(\sqrt{f^N(\eta_x^i)}-\sqrt{f^N(\eta)}\Big)d\nu_{\widehat{\rho}}^N(\eta)\\
&= \sum_{x\in\mathbb{T}_N^d}\sum_{i=0}^{3}\int\widetilde{c}(x,\eta,i)\Big(\sqrt{f^N(\eta)}\sqrt{f^N(\eta_x^i)}-f^N(\eta)\Big)d\nu_{\widehat{\rho}}^N(\eta)\\
&\leqslant \frac{1}{4}\sum_{x\in\mathbb{T}_N^d}\sum_{i=0}^{3}\int\widetilde{c}(x,\eta,i)f^N(\eta)d\nu_{\widehat{\rho}}^N(\eta)+\sum_{x\in\mathbb{T}_N^d}\sum_{i}\int\widetilde{c}(x,\eta,i)\Big(f^N(\eta_x^i)-f^N(\eta)\Big)d\nu_{\widehat{\rho}}^N(\eta),
\end{aligned}$$

using inequality $AB \leqslant \frac{1}{2a}A^2 + \frac{a}{2}B^2$ for $A, B, a > 0$ for the last bound. We deduce an estimate by Cauchy-Schwarz inequality to bound the first integral by the L^2-norm of f^N while the second integral is null since $\nu_{\widehat{\rho}}^N$ is invariant with respect to the auxiliary generator $\mathcal{L}_N^R$. □

4.B Properties of measures

Recall the measure we defined on $\mathbb{T}_N^d$ by $\overline{\nu}_{\widehat{\psi}}^N$ (4.2.6) for any vector $\widehat{\psi} = (\psi_0, \psi_1, \psi_2, \psi_3) \in \mathbb{R}^4$:

$$\overline{\nu}_{\widehat{\psi}}^N(\eta) := \prod_{x \in \mathbb{T}_N^d} \frac{1}{Z_{\widehat{\psi}}} \exp\left(\sum_{i=0}^{3} \psi_i \mathbf{1}\{\eta(x) = i\}\right) \tag{4.B.1}$$

where $Z_{\widehat{\psi}} = \sum_{i=0}^{3} \exp(\psi_i)$ is the normalization constant. Using that $\mathbf{1}\{\eta(x) = 0\} = 1 - \sum_{i=1}^{3} \mathbf{1}\{\eta(x) = i\}$, fix $\bar{\psi}_k = \psi_k - \psi_0$ for $1 \leqslant k \leqslant 3$ so that

$$\overline{\nu}_{(\overline{\psi}_1, \overline{\psi}_2, \overline{\psi}_3)}^N(\eta) = \prod_{x \in \mathbb{T}_N^d} \frac{\exp\left(\sum_{i=1}^{3} \bar{\psi}_i \mathbf{1}\{\eta(x) = i\}\right)}{1 + \sum_{i=1}^{3} \exp(\overline{\psi}_i)}.$$

To parametrize the invariant measure by the density of each type of particles, first deal with a change of variables as follows. Denote by $R(\cdot)$ the expectation of each occupation variable of a site x by type i under $\overline{\nu}_{\widehat{\psi}}^N$,

$$R : \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} \longmapsto \frac{1}{Z_{\widehat{\psi}}} \begin{pmatrix} \exp(\overline{\psi}_1) \\ \exp(\overline{\psi}_2) \\ \exp(\overline{\psi}_3) \end{pmatrix}.$$

Let the vector of densities $\widehat{\rho} = (\rho_1, \rho_2, \rho_3)$ such that $\rho_i \in [0,1]$ and $\rho_1 + \rho_2 + \rho_3 = 1 - \rho_0$. Then for all $i = 0, 1, 2, 3$, ρ_i satisfies

$$\overline{\nu}_{\widehat{\psi}}^N(\eta(x) = i) = \frac{1}{Z_{\widehat{\psi}}} \exp(\psi_i) = \rho_i. \tag{4.B.2}$$

Proposition 4.B.1. *The vector $\widehat{\rho}$ such that $1 - \rho_0 = \rho_1 + \rho_2 + \rho_3$ is uniquely determined by the vector $\widehat{\psi}$.*

Proof. Since we parametrize the measures by $\widehat{\rho}$, for all i :

$$\frac{1}{1 + \sum_{i=1}^{3} \exp(\overline{\psi}_i)} \exp(\overline{\psi}_i) = \rho_i.$$

And $(\overline{\psi}_1, \overline{\psi}_2, \overline{\psi}_3)$ solves the following system of equations

$$\begin{cases} \exp(\overline{\psi}_1) &= \rho_1(1 + \exp(\overline{\psi}_1) + \exp(\overline{\psi}_2) + \exp(\overline{\psi}_3)) \\ \exp(\overline{\psi}_2) &= \rho_2(1 + \exp(\overline{\psi}_1) + \exp(\overline{\psi}_2) + \exp(\overline{\psi}_3)) \\ \exp(\overline{\psi}_3) &= \rho_3(1 + \exp(\overline{\psi}_1) + \exp(\overline{\psi}_2) + \exp(\overline{\psi}_3)) \end{cases}$$

which can be rewritten as

$$\begin{cases} \overline{\psi}_1 &= \log(\dfrac{\rho_1}{\rho_0}) \\ \overline{\psi}_2 &= \log(\dfrac{\rho_2}{\rho_0}) \\ \overline{\psi}_3 &= \log(\dfrac{\rho_3}{\rho_0}) \end{cases}$$

One gets a triplet (ρ_1, ρ_2, ρ_3) such that $1 = \rho_0 + \rho_1 + \rho_2 + \rho_3$, by the transformation Ψ :

$$\Psi : \begin{pmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \end{pmatrix} \longmapsto \begin{pmatrix} \log\left(\dfrac{\rho_1}{1-\rho_1-\rho_2-\rho_3}\right) \\ \log\left(\dfrac{\rho_2}{1-\rho_1-\rho_2-\rho_3}\right) \\ \log\left(\dfrac{\rho_3}{1-\rho_1-\rho_2-\rho_3}\right) \end{pmatrix}. \tag{4.B.3}$$

where Ψ is the inverse function of R. $\square$

One can hence define uniquely a new product measure parametrize by the triplet $\widehat{\rho} = (\rho_1, \rho_2, \rho_3)$ by :

$$\nu^N_{\widehat{\rho}}(.) := \bar{\nu}^N_{\Psi(\rho_1,\rho_2,\rho_3)}(.) \tag{4.B.4}$$

One gets a family of measures whose marginal is given by $\nu^N_{\widehat{\rho}}(\eta(x) = k) = \rho_k$. In particular,

$$\nu^N_{\widehat{\rho}}(\eta(x) = 0) = 1 - \rho_1 - \rho_2 - \rho_3$$

Lemma 4.B.1. *The measure $\nu^N_{\widehat{\rho}}$ is reversible with respect to the generator of rapid-stirring process.*

Proof. Let $(\zeta_t)_{t\geqslant 0}$ be a stirring process with generator $\mathcal{L}^D_N$ on $\{0,1,2,3\}^{\mathbb{T}^d_N}$. For any cylinder function f, by posing $\xi = \zeta^{x,y}$:

$$\begin{aligned} \int \mathcal{L}^D f(\zeta) d\nu^N_{\widehat{\rho}}(\zeta) &= \int \sum_{\substack{x,y\in\mathbb{T}^d_N \\ \|x-y\|=1}} \Big(f(\zeta^{x,y}) - f(\zeta)\Big) d\nu^N_{\widehat{\rho}}(\zeta) \\ &= \int \sum_{\substack{x,y\in\mathbb{T}^d_N \\ \|x-y\|=1}} f(\zeta^{x,y}) d\nu^N_{\widehat{\rho}}(\zeta) - \int f(\zeta) d\nu^N_{\widehat{\rho}}(\zeta) \\ &= \int \sum_{\substack{x,y\in\mathbb{T}^d_N \\ \|x-y\|=1}} f(\xi) \frac{\nu^N_{\widehat{\rho}}(\xi^{y,x})}{\nu^N_{\widehat{\rho}}(\xi)} d\nu^N_{\widehat{\rho}}(\xi) - \int f(\zeta) d\nu^N_{\widehat{\rho}}(\zeta), \end{aligned}$$

and since (4.2.6) is product, $\dfrac{\nu^N_{\widehat{\rho}}(\zeta^{x,y})}{\nu^N_{\widehat{\rho}}(\xi^{y,x})} = 1.$ $\square$

A useful formula of change of variables :

Lemma 4.B.2. *Let $i, j \in \{0,1,2,3\}$ such that $i \neq j$. For any cylinder functions f, g and $\alpha > 0$,*

$$\int \alpha f(\eta_x^i) g(\eta) \mathbf{1}\{\eta(x) = j\} d\nu_{\hat{\rho}}^N(\eta) = \int \alpha \frac{\rho_j}{\rho_i} f(\eta) g(\eta_x^j) \mathbf{1}\{\eta(x) = i\} d\nu_{\hat{\rho}}^N(\eta) \tag{4.B.5}$$

Proof. Pose $\xi = \eta_x^i$,

$$\begin{aligned} &\int \alpha f(\eta_x^i) g(\eta) \mathbf{1}\{\eta(x) = j\} d\nu_{\hat{\rho}}^N(\eta) \\ &\quad = \int \alpha f(\xi) g(\xi_x^j) \mathbf{1}\{\xi(x) = i\} \frac{\nu_{\hat{\rho}}^N(\xi_x^j(x) = j)}{\nu_{\hat{\rho}}^N(\xi(x) = i)} d\nu_{\hat{\rho}}^N(\xi) \\ &\quad = \int \alpha f(\xi) g(\xi_x^j) \mathbf{1}\{\xi(x) = i\} \frac{\rho_j}{\rho_i} d\nu_{\hat{\rho}}^N(\xi) \end{aligned}$$

□

Define a generator $\mathcal{L}_N$ by

$$\mathcal{L}_N = \sum_{i=0}^{3} c(x, \eta, i) \Big(f(\eta_x^i) - f(\eta) \Big) \tag{4.B.6}$$

where for positive $\alpha, \beta, \gamma, \kappa, \alpha_1, \alpha_2, \beta_1, \beta_2$:

$$\begin{aligned} c(x,\eta,0) &= \begin{cases} \alpha_1 \text{ if } \eta(x) = 1 \\ \alpha_2 \text{ if } \eta(x) = 2 \end{cases} \quad c(x,\eta,1) = \begin{cases} \alpha \text{ if } \eta(x) = 0 \\ \alpha_2 \text{ if } \eta(x) = 3 \end{cases} \\ c(x,\eta,2) &= \begin{cases} r \text{ if } \eta(x) = 0 \\ \alpha_1 \text{ if } \eta(x) = 3 \end{cases} \quad c(x,\eta,3) = \begin{cases} r \text{ if } \eta(x) = 1 \\ \gamma \text{ if } \eta(x) = 2 \end{cases} \end{aligned} \tag{4.B.7}$$

Lemma 4.B.3. *Let $(\mathcal{L}_N)^\star$ be the adjoint of $\mathcal{L}_N$ in $L^2(\nu_{\hat{\rho}}^N)$, then $(\mathcal{L}_N)^\star$ is given for any cylinder function g on E_N by :*

$$\begin{aligned} (\mathcal{L}_N)^\star g(\eta) = \sum_{x \in \mathbb{T}_N^d} \Bigg\{ &(\alpha_1 \frac{\rho_1}{\rho_0} g(\eta_x^1) - \alpha g(\eta) + \alpha_2 \frac{\rho_2}{\rho_0} g(\eta_x^2) - r g(\eta)) \mathbf{1}_{\{\eta(x)=0\}} \\ &+ (\alpha \frac{\rho_0}{\rho_1} g(\eta_x^0) - \alpha_1 g(\eta) + \alpha_2 \frac{\rho_3}{\rho_1} g(\eta_x^3) - r g(\eta)) \mathbf{1}_{\{\eta(x)=1\}} \\ &+ (r \frac{\rho_0}{\rho_2} g(\eta_x^0) - \alpha_2 g(\eta) + \alpha_1 \frac{\rho_3}{\rho_2} g(\eta_x^3) - \gamma g(\eta)) \mathbf{1}_{\{\eta(x)=2\}} \\ &+ (r \frac{\rho_1}{\rho_3} g(\eta_x^1) - \alpha_2 g(\eta) + \gamma \frac{\rho_2}{\rho_3} g(\eta_x^2) - \alpha_1 g(\eta)) \mathbf{1}_{\{\eta(x)=3\}} \Bigg\} \\ =: \sum_{x \in \mathbb{T}_N^d} \sum_{i=0}^{3} c^\star(x, \eta, i) [g(\eta^i) - g(\eta)] \end{aligned}$$

Proof.

$$\int g(\eta)\mathcal{L}_N f(\eta)d\nu^N_{\hat\rho}(\eta) = \sum_{x\in\mathbb{T}^d_N}\int\Bigg\{\Big(\alpha\big(f(\eta^1_x)-f(\eta)\big)+r\big(f(\eta^2_x)-f(\eta)\big)\Big)\mathbf{1}_{\{\eta(x)=0\}}$$
$$+\Big(\alpha_1\big(f(\eta^0_x)-f(\eta)\big)+r\big(f(\eta^3_x)-f(\eta)\big)\Big)\mathbf{1}_{\{\eta(x)=1\}}+\Big(\alpha_2\big(f(\eta^0_x)-f(\eta)\big)+\gamma\big(f(\eta^3_x)$$
$$-f(\eta)\big)\Big)\mathbf{1}_{\{\eta(x)=2\}}+\Big(\alpha_2\big(f(\eta^1_x)-f(\eta)\big)+\alpha_1\big(f(\eta^2_x)-f(\eta)\big)\Big)\mathbf{1}_{\{\eta(x)=3\}}\Bigg\}\cdot g(\eta)d\nu^N_{\hat\rho}(\eta)$$
$$=\sum_{x\in\mathbb{T}^d_N}\int\Bigg\{\Big(\alpha_1\frac{\rho_1}{\rho_0}g(\eta^1_x)-\alpha g(\eta)+\alpha_2\frac{\rho_2}{\rho_0}g(\eta^2_x)-rg(\eta)\Big)\mathbf{1}_{\{\eta(x)=0\}}+\Big(\alpha\frac{\rho_0}{\rho_1}g(\eta^0_x)-\alpha_1 g(\eta)$$
$$+\alpha_2\frac{\rho_3}{\rho_1}g(\eta^3_x)-rg(\eta)\Big)\mathbf{1}_{\{\eta(x)=1\}}+\Big(r\frac{\rho_0}{\rho_2}g(\eta^0_x)-\alpha_2 g(\eta)+\alpha_1\frac{\rho_3}{\rho_2}g(\eta^3_x)-\gamma g(\eta)\Big)\mathbf{1}_{\{\eta(x)=2\}}$$
$$+\Big(r\frac{\rho_1}{\rho_3}g(\eta^1_x)-\alpha_2 g(\eta)+\gamma\frac{\rho_2}{\rho_3}g(\eta^2_x)-\alpha_1 g(\eta)\Big)\mathbf{1}_{\{\eta(x)=3\}}\Bigg\}\cdot f(\eta)d\nu^N_{\hat\rho}(\eta)$$
$$=\int f(\eta)(\mathcal{L}_N)^\star g(\eta)d\nu^N_{\hat\rho}(\eta)$$

□

4.C Quadratic variations computations

We prove in this section computations of the quadratic variation (4.3.11) of the martingale $M^{N,i}_t$ defined in 4.3.1, for $i=1,2,3$.

Lemma 4.C.1.

$$\langle M^{N,i}\rangle_t = \frac{N^2}{2N^{2d}}\int_0^t\sum_{x\in\mathbb{T}^d_N}\sum_{z\neq x,\|z-x\|=1}\Big(G_i(z/N)-G_i(x/N)\Big)^2\Big(\eta_{i,s}(z)-\eta_{i,s}(x)\Big)^2ds$$
$$+\frac{1}{N^{2d}}\int_0^t\sum_{x\in\mathbb{T}^d_N}G_i^2(x/N)\Big(1-2\eta_{i,s}(x)\Big)\mathcal{L}^R_N\eta_{i,s}(x)ds \quad (4.C.1)$$

Proof. The quadratic variation of $M^{N,i}_t$ is given, for any function $\widehat{G}\in\mathcal{C}(\mathbb{T}^d;\mathbb{R}^3)$, by

$$\langle M^{N,i}\rangle_t=\int_0^t\Big\{\mathfrak{L}_N\langle\pi^{N,i}_s,G_i\rangle^2-2\langle\pi^{N,i}_s,G_i\rangle\mathfrak{L}_N\langle\pi^{N,i}_s,G_i\rangle\Big\}ds$$

We shall prove the two following equalities :

$$N^2\int_0^t\Big\{\mathcal{L}^D_N\langle\pi^{N,i}_s,G_i\rangle^2-2\langle\pi^{N,i}_s,G_i\rangle\mathcal{L}^D_N\langle\pi^{N,i}_s,G_i\rangle\Big\}ds$$

$$= \frac{N^2}{2N^{2d}} \int_0^t \sum_{x\in\mathbb{T}_N^d} \sum_{z\neq x, \|z-x\|=1} \Big(G_i(z/N) - G_i(x/N)\Big)^2 \Big(\eta_{i,s}(z) - \eta_{i,s}(x)\Big)^2 ds \quad (4.C.2)$$

$$\int_0^t \Big\{\mathcal{L}_N^R \langle \pi_s^{N,i}, G_i\rangle^2 - 2\langle \pi_s^{N,i}, G_i\rangle \mathcal{L}_N^R \langle \pi_s^{N,i}, G_i\rangle\Big\} ds$$

$$= \frac{1}{N^{2d}} \int_0^t \sum_{x\in\mathbb{T}_N^d} G_i^2(x/N)\Big(1 - 2\eta_{i,s}(x)\Big)\mathcal{L}_N^R \eta_{i,s}(x) ds \quad (4.C.3)$$

Let us prove first (4.C.2).

$$\mathcal{L}_N^D \langle \pi_s^{N,i}, G_i\rangle^2 = \frac{1}{N^{2d}} \sum_{x\in\mathbb{T}_N^d} G_i^2(x/N)\mathcal{L}_N \eta_{i,s}(x)$$

$$+ \frac{1}{N^{2d}} \sum_{\substack{x,y\in\mathbb{T}_N^d \\ x\neq y, \|x-y\|>1}} G_i(x/N)G_i(y/N)\Big(\eta_{i,s}(x)\mathcal{L}_N^D \eta_{i,s}(y) + \eta_{i,s}(y)\mathcal{L}_N^D \eta_{i,s}(x)\Big)$$

$$+ \frac{1}{N^{2d}} \sum_{\substack{x,y\in\mathbb{T}_N^d \\ x\neq y, \|x-y\|=1}} G_i(x/N)G_i(y/N)\Bigg\{ \sum_{\substack{z\in\mathbb{T}_N^d \\ z\neq y, |z-x|=1}} \Big(\eta_{i,s}(z)\eta_{i,s}(y) - \eta_{i,s}(x)\eta_{i,s}(y)\Big)$$

$$+ \sum_{\substack{u\in\mathbb{T}_N^d \\ u\neq y, \|u-x\|=1}} \Big(\eta_{i,s}(u)\eta_{i,s}(x) - \eta_{i,s}(x)\eta_{i,s}(y)\Big)\Bigg\}$$

and

$$-2\langle \pi_s^{N,i}, G_i\rangle \mathcal{L}_N^D \langle \pi_s^{N,i}, G_i\rangle = \frac{-2}{N^{2d}} \sum_{x\in\mathbb{T}_N^d} G_i^2(x/N)\eta_{i,s}(x)\mathcal{L}_N^D \eta_{i,s}(x)$$

$$- \frac{1}{N^{2d}} \sum_{\substack{x,y\in\mathbb{T}_N^d \\ x\neq y}} G_i(x/N)G_i(y/N) \cdot \Bigg\{ \sum_{\substack{z\in\mathbb{T}_N^d \\ z\neq y, \|z-x\|=1}} \Big(\eta_{i,s}(z)\eta_{i,s}(y) - \eta_{i,s}(x)\eta_{i,s}(y)\Big)$$

$$+ \eta_{i,s}(y) - \eta_{i,s}(y)\eta_{i,s}(x)$$

$$+ \sum_{\substack{u\in\mathbb{T}_N^d \\ u\neq y, \|u-x\|=1}} \Big(\eta_{i,s}(u)\eta_{i,s}(x) - \eta_{i,s}(x)\eta_{i,s}(y)\Big) + \eta_{i,s}(x) - \eta_{i,s}(y)\eta_{i,s}(x)\Bigg\}$$

$$- \frac{1}{N^{2d}} \sum_{\substack{x,y\in\mathbb{T}_N^d \\ x\neq y, \|x-y\|>1}} G_i(x/N)G_i(y/N)\Big(\eta_{i,s}(x)\mathcal{L}_N^D \eta_{i,s}(y) + \eta_{i,s}(y)\mathcal{L}_N^D \eta_{i,s}(x)\Big)$$

so that,

$$N^2 \int_0^t \Bigg\{\mathcal{L}_N^D \langle \pi_s^{N,i}, G_i\rangle^2 - 2\langle \pi_s^{N,i}, G_i\rangle \mathcal{L}_N^D \langle \pi_s^{N,i}, G_i\rangle\Bigg\} ds$$

$$= \frac{N^2}{N^{2d}} \int_0^t \sum_{x \in \mathbb{T}_N^d} \sum_{z : \|z-x\|=1} G_i^2(x/N)\Big(\eta_{i,s}(z) - \eta_{i,s}(x)\Big)ds$$
$$- \frac{N^2}{N^{2d}} \int_0^t \sum_{x \in \mathbb{T}_N^d} \sum_{z : \|z-x\|=1} G_i^2(x/N)\Big(2\eta_{i,s}(z)\eta_{i,s}(x) - \eta_{i,s}(x)\Big)ds$$
$$- \frac{N^2}{N^{2d}} \int_0^t \sum_{x \in \mathbb{T}_N^d} \sum_{z \neq x, \|z-x\|=1} G_i(x/N)G_i(z/N)\Big(\eta_{i,s}(z) - 2\eta_{i,s}(z)\eta_{i,s}(x) + \eta_{i,s}(x)\Big)ds$$
$$= \frac{N^2}{2N^{2d}} \int_0^t \sum_{x \in \mathbb{T}_N^d} \sum_{z \neq x, \|z-x\|=1} \Big(G_i(z/N) - G_i(x/N)\Big)^2\Big(\eta_{i,s}(z) - \eta_{i,s}(x)\Big)^2 ds$$

On the other hand,

$$\int_0^t \Big\{\mathcal{L}_N^R \langle \pi_s^{N,i}, G_i\rangle^2 - 2\langle \pi_s^{N,i}, G_i\rangle \mathcal{L}_N^R \langle \pi_s^{N,i}, G_i\rangle\Big\}ds$$
$$= \frac{1}{N^{2d}} \int_0^t \sum_{x \in \mathbb{T}_N^d} G_i^2(x/N)\mathcal{L}_N^R \eta_{i,s}(x)ds \tag{4.C.4}$$
$$+ \frac{1}{N^{2d}} \int_0^t \sum_{y \neq x} G_i(x/N)G_i(y/N)\Big(\eta_{i,s}(x)\mathcal{L}_N^R\eta_{i,s}(y) + \eta_{i,s}(y)\mathcal{L}_N^R\eta_{i,s}(x)\Big)ds$$
$$- \frac{2}{N^{2d}} \int_0^t \sum_{x \in \mathbb{T}_N^d} G_i^2(x/N)\eta_{i,s}(x)\mathcal{L}_N^R\eta_{i,s}(x)ds$$
$$- \frac{1}{N^{2d}} \int_0^t \sum_{y \neq x} G_i(x/N)G_i(y/N)\Big(\eta_{i,s}(x)\mathcal{L}_N^R\eta_{i,s}(y) + \eta_{i,s}(y)\mathcal{L}_N^R\eta_{i,s}(x)\Big)ds$$
$$= \frac{1}{N^{2d}} \int_0^t \sum_{x \in \mathbb{T}_N^d} G_i^2(x/N)\Big(1 - 2\eta_{i,s}(x)\Big)\mathcal{L}_N^R\eta_{i,s}(x)ds \tag{4.C.5}$$

Using 4.3.2, we have for each $i = 1, 2, 3$:

$$\Big(1 - 2\eta_1(x)\Big)\mathcal{L}_N^R\eta_1(x)$$
$$= \mathcal{L}_N^R\eta_1(x) + 2\Big\{(r+1)\eta_1(x) + \Big(\lambda_1 \sum_{y:\|y-x\|=1} \eta_1(y) + \lambda_2 \sum_{y:\|y-x\|=1} \eta_3(y)\Big)\eta_1(x)\Big\}$$
$$= \Big(\lambda_1 \sum_{y:\|y-x\|=1} \eta_1(y) + \lambda_2 \sum_{y:\|y-x\|=1} \eta_3(y)\Big)(1 + \eta_1(x) - \eta_2(x) - \eta_3(x)) + \eta_3(x)$$
$$+ (r+1)\eta_1(x)$$
$$\Big(1 - 2\eta_2(x)\Big)\mathcal{L}_N^R\eta_2(x) = \mathcal{L}_N^R\eta_{2,s}(x) + 2\eta_2(x) = r\eta_0(x) + \eta_3(x) + \eta_2(x)$$
$$\Big(1 - 2\eta_3(x)\Big)\mathcal{L}_N^R\eta_3(x) = \mathcal{L}_N^R\eta_3(x) + 4\eta_3(x) = r\eta_1(x) + 2\eta_3(x).$$

Gathering all these estimates, one has

$$\begin{aligned}\int_0^t & \Big\{\mathcal{L}_N^R\langle \pi_s^{N,i}, G_i\rangle^2 - 2\langle \pi_s^{N,i}, G_i\rangle \mathcal{L}_N^R\langle \pi_s^{N,i}, G_i\rangle\Big\} ds \\ = & \quad \Big(\lambda_1 \sum_{y:\|y-x\|=1} \eta_1(y) + \lambda_2 \sum_{y:\|y-x\|=1} \eta_3(y)\Big)(\eta_0(x) + 2\eta_1(x)) \\ & \qquad + r\eta_0(x) + (2r+1)\eta_1(x) + \eta_2(x) + 4\eta_3(x)\end{aligned} \tag{4.C.6}$$

□

4.D Topology of the Skorohod space

We summarize here some useful tips concerning the Skorohod space, see [7, Chapter 3] for further details.

Fix $T > 0$. Recall $D([0,T], (\mathcal{M}_+^1)^3)$ stands for the set of right-continuous with left limits trajectories with values in $(\mathcal{M}_+^1)^3$, endowed with the Skorohod topology and equipped with its Borel $\sigma-$ algebra.

Define a metric on $\mathcal{M}_+^1$ by introducing for every dense sequence of continuous functions $\{f_k, k \geqslant 1\}$ on $\mathbb{T}^d$ the distance $\delta(\cdot,\cdot)$ by

$$\delta(\mu, \nu) = \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{|\langle \mu, f_k\rangle - \langle \nu, f_k\rangle|}{1 + |\langle \mu, f_k\rangle - \langle \nu, f_k\rangle|} \tag{4.D.1}$$

The space $\mathcal{M}^+$ is complete with respect to the endowed weak topology, and any set $A \subset \mathcal{M}_+^1$ is relatively compact in $\mathcal{M}_+^1$ if and only if

$$\sup_{\mu \in A} \langle \mu, 1\rangle < \infty$$

Let $\mathcal{E}$ be a polish space equipped with the metric $\delta(\cdot,\cdot)$ and consider a sequence of probability measures $(P^N)_N$ in $D([0,T],\mathcal{E})$. Let Λ be the set of increasing continuous functions on $[0,T]$. Define,

$$\text{for all } \lambda \in \Lambda,\ \|\lambda\| = \sup_{s \neq t} \left| \frac{\lambda(t) - \lambda(s)}{t-s} \right|$$

and

$$d(\mu,\nu) := \inf_{\lambda \in \Lambda} \left\{ \|\lambda\| \vee \sup_{0 \leqslant t \leqslant T} \delta(\mu_t, \nu_{\lambda(t)}) \right\}$$

Proposition 4.D.1. *The space $D([0,T],\mathcal{E})$ equipped with the metric $\delta(\cdot,\cdot)$ is polish.*

To extend Ascoli's theorem to the space $D([0,T],\mathcal{E})$, one introduces the modulus of continuity :

$$\omega_\mu(\gamma) = \sup_{|t-s| \leqslant \gamma} \delta(\mu_s, \mu_t) \tag{4.D.2}$$

A continuous function on $[0,T]$ is uniformly continuous. To get something similar for functions in the Skorohod space, introduce

Lemma 4.D.1. *For all $\mu \in D([0,T],\mathcal{E})$ and $\epsilon > 0$, there exists a sequence of times $\{t_i\}_{0\leqslant i\leqslant r}$ such that*

$$0 = t_0 < t_1 < ... < t_r = T \quad \textit{and} \quad \omega_\mu(t_i - t_{i-1}) > \epsilon,\ i = 1, ..., r.$$

For such a sequence $\{t_i\}_{0\leqslant i\leqslant r}$, one can define the modified modulus of continuity by

$$\omega'_\mu(\gamma) = \inf_{\{t_i\}_{0\leqslant i\leqslant r}} \max_{0\leqslant i\leqslant r} \sup_{t_i\leqslant s<t<t_{i+1}} \delta(\mu_s,\mu_t). \tag{4.D.3}$$

One can characterize the compact sets of $D([0,T],\mathcal{E})$ thanks the modified modulus of continuity :

Proposition 4.D.2. *A set A in $D([0,T],\mathcal{E})$ is relatively compact if and only if*

(1) $\{\mu_t : \mu \in A, t \in [0,T]\}$ is relativement compact on $\mathcal{E}$.

(2) $\lim_{\gamma\to 0} \sup_{\mu\in A} \omega'_\mu(\gamma) = 0$.

One can now state Prohorov's theorem,

Theorem 4.D.1. *Let $\{P^N, N \geqslant 1\}$ be a sequence of probability measures in $D([0,T],\mathcal{E})$. Then $\{P^N, N \geqslant 1\}$ is relatively compact if and only if*

(1) For all $t \in [0,T]$ and $\epsilon > 0$, there exists a compact set $K(t,\epsilon) \subset \mathcal{E}$ such that $\sup_{N\geqslant 1} P^N(\mu_t \in K(t,\epsilon)^c) \leqslant \epsilon$.

(2) For any $\epsilon > 0$, $\lim_{\gamma\to 0} \overline{\lim}_{N\to\infty} P^N(\mu : \omega'_\mu(\gamma) > \epsilon) = 0$.

On the other hand, condition (2) can by substituted by the following sufficient condition :

Proposition 4.D.3 (D. Aldous (1978))**.** *A sequence of probability measure $\{P^N, N \geqslant 1\}$ in $D([0,T],\mathcal{E})$ satisfies(2) of Theorem 4.D.1 if*

$$\lim_{\gamma\to 0} \overline{\lim}_{N\to\infty} \sup_{\substack{\tau\in\mathfrak{T}_T\\ \theta\leqslant\gamma}} P^N(\delta(\mu_\tau,\mu_{\tau+\theta}) > \epsilon) = 0 \tag{4.D.4}$$

where $\mathfrak{T}_T$ stands for the set of stopping times bounded from above by T.

For the space $\mathcal{M}_+^1$ endowed with the weak topology, to prove the relative compactness for a sequence of measures $(\mathbb{Q}^N_{\mu^N}, N \geqslant 1)$ defined in $\mathcal{D}([0,T],\mathcal{M}_+^1)$, it is enough to check Prohorov's theorem 4.D.1 for real-valued processes by projecting the empirical measures with functions of a dense countable set of $\mathcal{C}(\mathbb{T}^d;\mathbb{R})$:

Proposition 4.D.4. *Let $\{g_k, k \geqslant 1\}$ be a dense countable set in $\mathcal{C}(\mathbb{T}^d)$ with $g_1 = 1$. A sequence of probability measures $(\mathbb{Q}^N_{\mu^N})_{N\geqslant 1}$ is relatively compact in $D([0,T],\mathcal{M}_+^1)$ if for any positive integer k, the sequence $(\mathbb{Q}^N_{\mu^N} g_k^{-1})_{N\geqslant 1}$ in $D([0,T],\mathbb{R})$ defined by*

$$\mathbb{Q}^N_{\mu^N} g_k^{-1}(A) = \mathbb{Q}^N_{\mu^N}(\pi^{N,i} : \langle \pi^{N,i}, g_k\rangle \in A)$$

is relatively compact.

5

Hydrodynamic limits of a generalized contact process with stochastic reservoirs or in infinite volume

Contents

We consider a generalized contact process represented by a two species process evolving either in a bounded domain in contact with particles reservoirs at different densities, or in $\mathbb{Z}^d$. In both cases we study the law of large numbers for current and densities.

5.1 Introduction

In this chapter, we consider a generalized contact process describing the evolution on a lattice of three types of populations labeled respectively by 1, 2 and 3. This process was introduced in [49] (in preparation), see Chapters 2 and 3, to model the *sterile insect technique*, developed by E. Knipling and R. Bushland (see for instance [46, 27]) in the fifties to control the New World screw worm, a serious threat to warm-blooded animals. This pest has been eradicated from the USA and Mexico only in recent decades. The technique works as follows : Screw worms are reared in captivity and exposed to Gamma rays. The male screw worms become sterile. If a sufficient number of sterile males are released in the wild then enough female screw worms are mated by sterile males so that the number of offspring will decrease generation after generation. This technique is well suited for screw worms, because female apparently mate only once in their lifetime, but is also being tried for a large variety of pests, including a current project to fight dengue in Brazil.

The particle system $(\eta_t)_{t\geqslant 0}$ we look at has state space $\{0,1,2,3\}^S$, where $S \subset \mathbb{Z}^d$, typically $d = 2$. Each site of S is either empty (we say it is in state 0), occupied by wild screw worms only (state 1), by sterile screw worms only (state 2), or by wild and sterile screw worms together (state 3). We keep track only of the presence or not of the type of the male screw worms (and not of their number), and we assume that enough female are around as not to limit mating. A site gets sterile males at rate r independently of everything else (this corresponds to the artificial introduction of sterile males). The birth rate is 0 at sites in state 2, λ_1 at sites in state 1, and λ_2 at sites in state 3. We assume that $\lambda_2 < \lambda_1$ to reflect the fact that at sites in state 3 the fertility is decreased. Deaths for each population occur at all sites at rate 1, being mutually independent.

If η denotes a current configuration, the transitional mechanism for the generalized contact dynamics at a site x can be summarized as follows :

$$\begin{array}{llll} 0 \to 1 & \text{at rate } \lambda_1 n_1(x,\eta) + \lambda_2 n_3(x,\eta) & 1 \to 0 & \text{at rate } 1 \\ 0 \to 2 & \text{at rate } r & 2 \to 0 & \text{at rate } 1 \\ 1 \to 3 & \text{at rate } r & 3 \to 1 & \text{at rate } 1 \\ 2 \to 3 & \text{at rate } \lambda_1 n_1(x,\eta) + \lambda_2 n_3(x,\eta) & 3 \to 2 & \text{at rate } 1 \end{array} \tag{5.1.1}$$

where $n_i(x,\eta)$ is the number of nearest neighbors of x in state i for $i = 1,3$. This dynamics has been studied in $S = \mathbb{Z}^d$ in [49], see Chapter 2, where a phase transition

This chapter is a joint work with M. Mourragui and E. Saada [50].

in r was exhibited : Assuming that $\lambda_2 \leqslant \lambda_c < \lambda_1$, where λ_c denotes the critical value of the d-dimensional basic contact process, there exists a critical value r_c such that the populations in states 1 and 3 survive for $r < r_c$, and die out for $r \geqslant r_c$.

Our goal in the present chapter is to add to the previous contact dynamics displacements of populations within S infinite volume case, as well, in the finite volume case, as departures from S and immigrations to S. We are interested in the evolution of the empirical densities of the 3 types of populations, for which we establish hydrodynamic limits. The limiting equations are given by a system of non-linear reaction-diffusion equations, with additionally Dirichlet boundary conditions.

More precisely, denote by $\mathbb{T}_N^{d-1}$ the $(d-1)$-dimensional microscopic torus of length N, where N is a scaling parameter. The non-conservative system that we consider evolves either in a bounded cylinder $\Lambda_N = \{-N, \cdots, N\} \times \mathbb{T}_N^{d-1}$ or in $\mathbb{Z}^d$. The cylinder Λ_N has length $2N + 1$ along the axis of direction e_1, where $(e_1, \ldots, e_d)$ denotes the canonical basis of $\mathbb{R}^d$.

In the bulk of Λ_N, resp. in $\mathbb{Z}^d$, particles evolve according to the superposition of an exchange dynamics representing the displacements of the populations in different states, and the above generalized contact process. In Λ_N, the movements of populations at the boundary Γ_N of the domain Λ_N are modelled thanks to reservoirs from which populations in different states are created or annihilated.

The exchange of the occupation variable $\eta(x)$ in any site x with the one of a nearest neighbour site is performed with rate 1. This exchange dynamics satisfies a detailed balance condition with respect to a family of Gibbs measures, parametrized by the so-called chemical potential $\widehat{\rho} = (\rho_1, \rho_2, \rho_3) \in \mathbb{R}^3$.

In the finite volume case, the reservoirs are modelled by a reversible generalized contact process with fixed density. More precisely, for a fixed smooth vector valued function $\widehat{b}(\cdot) = (b_1(\cdot), b_2(\cdot), b_3(\cdot))$ defined on the boundary of the domain, the rates of this contact process are chosen so that a Gibbs measure of varying chemical potential $\widehat{b}(\cdot)$ is reversible for it.

To deal with infinite volume, we establish bounds on the entropy production and on the Dirichlet forms valid for a boundary driven version of our process on $\Lambda_N^\infty = \{-N, ..., N\} \times \mathbb{Z}^{d-1}$, hence on $\mathbb{Z}^d$. We also establish uniqueness of the weak solution to the system of equations corresponding to the boundary driven case in infinite volume. The same method gives uniqueness on $\mathbb{Z}^d$.

In Section 5.2, we detail our model, and state our results, namely on the specific entropy (Theorem 5.2.1), the hydrodynamic limit of the boundary driven generalized process (Theorem 5.2.2), the hydrodynamic limit of the generalized process in $\mathbb{Z}^d$ (Theorem 5.2.3), a law of large numbers for currents (Proposition 5.2.1), uniqueness results for the equations in Subsection 5.2.6.

In Section 5.3, we prove Theorem 5.2.1 , in Section 5.4 we prove Theorem 5.2.2, in Section 5.5 we prove Proposition 5.2.1, in Section 5.6 we prove Theorem 5.2.3, results on uniqueness of solutions are proved in Section 5.7 and finally Appendices 5.A-5.B-5.C contain useful computations.

5.2 Notation and Results

5.2.1 The model

Instead of studying the three different values $\eta(x) = 1, 2, 3$ considered above, we introduce another interpretation for the model. The configuration space is now $\widehat{\Sigma}_N := \big(\{0,1\} \times \{0,1\}\big)^{\Lambda_N}$ or $\widehat{\Sigma} := (\{0,1\} \times \{0,1\})^{\mathbb{Z}^d}$; elements of $\widehat{\Sigma}_N$ (resp. $\widehat{\Sigma}$) are denoted by (ξ, ω). The correspondence with $(\eta_t)_{t\geqslant 0}$ is given by the following relations :

$$\begin{array}{lcl} \eta(x) = 0 & \Longleftrightarrow & (1-\xi(x))(1-\omega(x)) = 1\,, \\ \eta(x) = 1 & \Longleftrightarrow & \xi(x)(1-\omega(x)) = 1\,, \\ \eta(x) = 2 & \Longleftrightarrow & (1-\xi(x))\omega(x) = 1\,, \\ \eta(x) = 3 & \Longleftrightarrow & \xi(x)\omega(x) = 1\,. \end{array} \tag{5.2.1}$$

In other words, ξ-particles represent the wild screw worms, while ω-particles represent the sterile ones. On a site x, $\xi(x) = 1$ if wild screw worms are present on x, and $\omega(x) = 1$ if sterile screw worms are present on x. Both can be present, giving the state 3 for $\eta(x)$ or only one of them, giving the states 1 or 2 for $\eta(x)$.

The boundary driven generalized contact process with exchange of particles is the Markov process on $\widehat{\Sigma}_N$ whose generator $\mathfrak{L}_N := \mathfrak{L}_{\lambda_1,\lambda_2,r,\hat{b},N}$ can be decomposed as

$$\mathfrak{L}_N \;:=\; N^2\mathcal{L}_N + \mathbb{L}_N \;+ N^2\, L_{\hat{b},N}\,, \tag{5.2.2}$$

where $\mathcal{L}_N$ is the generator of exchanges of particles, $\mathbb{L}_N$ the generator of the generalized contact process, and $L_{\hat{b},N}$ the generator of the boundary dynamics. We now detail both dynamics and their properties.

For the exchange dynamics, the action of $\mathcal{L}_N$ on cylinder functions $f : \widehat{\Sigma}_N \to \mathbb{R}$ is

$$\mathcal{L}_N f(\xi,\omega) = \sum_{\substack{x,y\in\Lambda_N \\ \|x-y\|=1}} \left[f(\xi^{x,y},\omega^{x,y}) - f(\xi,\omega) \right] , \tag{5.2.3}$$

where for any $\xi \in \Sigma_N := \{0,1\}^{\Lambda_N}$, $\xi^{x,y}$ is the configuration obtained from $\xi \in \Sigma_N$, by exchanging the occupation variables $\xi(x)$ and $\xi(y)$, i.e.

$$(\xi^{x,y})(z) := \begin{cases} \xi(y) & \text{if } \; z = x\,, \\ \xi(x) & \text{if } \; z = y\,, \\ \xi(z) & \text{if } \; z \neq x, y\,. \end{cases}$$

Note that, since $(\xi,\omega) \in \widehat{\Sigma}_N$, these exchanges can be interpreted as jumps between sites x to y for ξ-particles and ω-particles, which do not influence each other.

To exhibit invariant measures for $\mathcal{L}_N$, for any $x \in \Lambda_N$, according to (5.2.1), we define

$$\begin{cases} \eta_1(x) = \xi(x)(1-\omega(x)) = \mathbf{1}_{\{\eta(x)=1\}}\,, \\ \eta_2(x) = (1-\xi(x))\omega(x) = \mathbf{1}_{\{\eta(x)=2\}}\,, \\ \eta_3(x) = \xi(x)\omega(x) = \mathbf{1}_{\{\eta(x)=3\}}\,. \end{cases} \tag{5.2.4}$$

By a misuse of language, when $\eta_i(x) = 1$ for $i = 1,2,3$, we say that there is a particle of type i at x.

The invariant measures will be product measures parametrized by three chemical potentials, since the exchange dynamics conserves the three quantities $\sum_{x\in\Lambda_N}\eta_i(x)$, $1 \leqslant i \leqslant 3$. It is convenient to complete (5.2.4) by defining, for $x \in \Lambda_N$,

$$\eta_0(x) = (1-\xi(x))(1-\omega(x)) = \mathbf{1}_{\{\eta(x)=0\}} = 1 - \eta_1(x) - \eta_2(x) - \eta_3(x)\,. \tag{5.2.5}$$

We denote by Λ the macroscopic open bounded cylinder $(-1,1)\times\mathbb{T}^{d-1}$ where $\mathbb{T}^k$ is the k-dimensional torus $[0,1)^k$. For a vector-valued function $\widehat{m} = (m_1,m_2,m_3) : \Lambda \to \mathbb{R}^3$, we define $\bar{\nu}^N_{\widehat{m}(\cdot)}$ as the product measure on Λ_N with varying chemical potential $\widehat{m}$,

$$d\bar{\nu}^N_{\widehat{m}(\cdot)}(\xi,\omega) = \widehat{Z}_{\widehat{m}}^{-1}\exp\Big\{\sum_{i=1}^{3}\sum_{x\in\Lambda_N} m_i(x/N)\eta_i(x)\Big\}, \tag{5.2.6}$$

where $\widehat{Z}_{\widehat{m}}$ is the normalization constant :

$$\widehat{Z}_{\widehat{m}} = \prod_{x\in\Lambda_N}\Big\{1 + \sum_{i=1}^{3}\exp(m_i(x/N))\Big\}. \tag{5.2.7}$$

Notice that the family of measures $\big\{\bar{\nu}^N_{\widehat{m}}\,,\ \ \widehat{m}\in\mathbb{R}^3\big\}$ with constant parameters is reversible with respect to the generator $\mathcal{L}_N$. For $\widehat{m}\in\mathbb{R}^3$ and $1\leqslant i\leqslant 3$, let $\psi_i(\widehat{m})$ be the expectation of $\eta_i(0)$ under $\bar{\nu}^N_{\widehat{m}}$:

$$\psi_i(\widehat{m}) \;=\; \mathbf{E}^{\bar{\nu}^N_{\widehat{m}}}\big[\eta_i(0)\big]\,.$$

Observe that the function $\widehat{\psi}$ defined on $(0,+\infty)^3$ by $\widehat{\psi}(\widehat{m}) = (\psi_1(\widehat{m}),\psi_2(\widehat{m}),\psi_3(\widehat{m}))$ is a bijection from $(0,+\infty)^3$ to $(0,1)^3$. We will therefore do a change of parameter : For every $\widehat{\rho} = (\rho_1,\rho_2,\rho_3)\in(0,1)^3$, we denote by $\nu^N_{\widehat{\rho}}$ the product measure such that

$$\rho_i \;=\; \mathbf{E}^{\nu^N_{\widehat{\rho}}}\big[\eta_i(0)\big]\,, \quad i = 1,2,3\,. \tag{5.2.8}$$

From now on, we work with the representation $\nu^N_{\widehat{\rho}(\cdot)}$ of the measure $\bar{\nu}^N_{\widehat{m}(\cdot)}$.

According to (5.1.1), the generator $\mathbb{L}_N := \mathbb{L}_{N,\lambda_1,\lambda_2,r}$ of the generalized contact process is given by

$$\begin{aligned}\mathbb{L}_N f(\xi,\omega) = &\sum_{x\in\Lambda_N}\Big(r(1-\omega(x))+\omega(x)\Big)\Big[f(\xi,\sigma^x\omega)-f(\xi,\omega)\Big]\\ &+\sum_{x\in\Lambda_N}\Big(\beta_N(x,\xi,\omega)\big(1-\xi(x)\big)+\xi(x)\Big)\Big[f(\sigma^x\xi,\omega)-f(\xi,\omega)\Big],\end{aligned} \tag{5.2.9}$$

with

$$\beta_N(x,\xi,\omega) = \lambda_1 \sum_{\substack{y\in\Lambda_N\\ \|y-x\|=1}} \xi(y)(1-\omega(y)) + \lambda_2 \sum_{\substack{y\in\Lambda_N\\ \|y-x\|=1}} \xi(y)\omega(y) \tag{5.2.10}$$

where $\|\cdot\|$ denotes the norm in $\mathbb{R}^d$, $\|u\| = \sqrt{\sum_{i=1}^d |u_i|^2}$, and for $\xi \in \Sigma_N$, $\sigma^x\xi$ is the configuration obtained from ξ by flipping the configuration at x, i.e.

$$(\sigma^x\xi)(z) := \begin{cases} 1-\xi(x) & \text{if } \; z = x\,,\\ \xi(z) & \text{if } \; z \neq x\,,\end{cases}$$

The representation (5.2.1) sheds light on the fact that (5.2.9) corresponds to a contact process (the ξ-particles) in a dynamic random environment, namely the ω-particles. Indeed, the ω-particles move by their own and are not influenced by ξ-particles, while ξ-particles have birth rates whose value depends on the presence or not of ω-particles. Note that in [49] (see Chapter 3) a variant of the generalized contact dynamics in a quenched random environment was also considered, with the (ξ,ω)-formalism. On the other hand, we noticed previously that ω-particles can also be considered as an environment for the exchange dynamics.

We now turn to the dynamics at the boundaries of the domain. We denote by $\overline{\Lambda} = [-1,1]\times\mathbb{T}^{d-1}$ the closure of Λ, and by $\Gamma = \partial\Lambda$ the boundary of $\Lambda : \Gamma = \{(u_1,\ldots,u_d) \in \overline{\Lambda} : u_1 = \pm 1\}$. For a metric space E, an any integer $1 \leqslant m \leqslant +\infty$ denote by $\mathcal{C}^m(\overline{\Lambda};E)$ (resp. $\mathcal{C}^m_c(\Lambda;E)$) the space of m-continuously differentiable functions on $\overline{\Lambda}$ with values in E (resp. with compact support in Λ) .

Fix a positive function $\widehat{b} : \Gamma \to \mathbb{R}^3_+$. Assume that there exists a neighbourhood V of $\overline{\Lambda}$ and a smooth function $\widehat{\theta} = (\theta_1,\theta_2,\theta_3) : V \to (0,1)^3$ in $\mathcal{C}^2(V;\mathbb{R}^3)$ such that

$$0 < c \leqslant \min_{1\leqslant i\leqslant 3} |\theta_i| \leqslant \max_{1\leqslant i\leqslant 3} |\theta_i| \leqslant C \leqslant 1 \tag{5.2.11}$$

for two positive constants c, C, and such that the restriction of $\widehat{\theta}$ to Γ is equal to $\widehat{b}$.

The boundary dynamics acts as a birth and death process on the boundary Γ_N of

Λ_N described by the generator $L_{\widehat{b},N}$ defined by

$$
\begin{aligned}
L_{\widehat{b},N} f(\xi,\omega) \;=\; & \sum_{x\in\Gamma_N} c_x\big(\widehat{b}(x/N),\xi,\sigma^x\omega\big)\Big[f(\xi,\sigma^x\omega)-f(\xi,\omega)\Big] \\
& + \sum_{x\in\Gamma_N} c_x\big(\widehat{b}(x/N),\sigma^x\xi,\omega\big)\big[f(\sigma^x\xi,\omega)-f(\xi,\omega)\big] \\
& + \sum_{x\in\Gamma_N} c_x\big(\widehat{b}(x/N),\sigma^x\xi,\sigma^x\omega\big)\big[f(\sigma^x\xi,\sigma^x\omega)-f(\xi,\omega)\big]\,,
\end{aligned}
\tag{5.2.12}
$$

where the rates $c_x\big(\widehat{b}(x/N),\xi,\omega\big)$ are given for $x\in\Gamma_N$ and $(\xi,\omega)\in\widehat{\Sigma}_N$ by

$$
c_x\big(\widehat{b}(x/N),\xi,\omega\big) \;=\; \sum_{i=0}^{3} b_i(x/N)\eta_i(x)\,, \tag{5.2.13}
$$

where $b_0(x/N) = 1-\sum_{i=1}^{3} b_i(x/N)$ and $\eta_i(x)$, $i=0,1,2,3$ are defined in (5.2.4)-(5.2.5). Using Lemma 5.A.2, note that the measure $\nu^N_{\widehat{\theta}}$ is reversible with respect to the generator $L_{\widehat{b},N}$.

As we deal with the process in infinite volume, define the generator in $\mathbb{Z}^d$ by omitting the subscript N in $\mathcal{L}_N$ and $\mathbb{L}_N$ to denote the sums are carried over $\mathbb{Z}^d$. In infinite volume, the process has generator : :

$$
\mathfrak{L} = N^2\mathcal{L} + \mathbb{L} \tag{5.2.14}
$$

Notice that in view of the diffusive scaling limit, the generator $\mathcal{L}_N$ (resp. $\mathcal{L}$) has been speeded up by N^2. We denote by $(\xi_t,\omega_t)_{t\geqslant 0}$ the Markov process on $\widehat{\Sigma}_N$ with generator $\mathfrak{L}_N$ (resp. on $\widehat{\Sigma}$ with generator $\mathfrak{L}$) and by $\mathbb{P}^{N,\widehat{b}}_{\mu}$ (resp. $\mathbb{P}^{N}_{\mu}$) its distribution if the initial configuration is distributed according to μ. Note that $\mathbb{P}^{N,\widehat{b}}_{\mu}$ (resp. $\mathbb{P}^{N}_{\mu}$) is a probability measure on the path space $D(\mathbb{R}_+,\widehat{\Sigma}_N)$ (resp. $D(\mathbb{R}_+,\widehat{\Sigma})$), which we consider endowed with the Skorohod topology and the corresponding Borel σ-algebra. Expectation with respect to $\mathbb{P}^{N,\widehat{b}}_{\mu}$ is denoted by $\mathbb{E}^{N,\widehat{b}}_{\mu}$ (resp. $\mathbb{E}^{N}_{\mu}$). We denote by $\mathcal{M}$ the space of finite signed measures either on Λ or $\mathbb{R}^d$, endowed with the weak topology. For a finite signed measure m and a continuous function F on Λ or $\mathbb{R}^d$, we let $\langle m,F\rangle$ be the integral of F with respect to m. For each configuration (ξ,ω), denote by $\widehat{\pi}^N = \widehat{\pi}^N(\xi,\omega) = (\pi^{N,1},\pi^{N,2},\pi^{N,3})\in\mathcal{M}^3$, where for $i=1,2,3$, the positive measure $\pi^{N,i}$ is obtained by assigning mass N^{-d} to each particle of type η_i :

$$
\pi^{N,i} \;=\; N^{-d}\sum_x \eta_i(x)\,\delta_{x/N}\,,
$$

where δ_u is the Dirac measure concentrated on u, and the sum is carried either on Λ_N or $\mathbb{Z}^d$. For any continuous function $\widehat{G}=(G_1,G_2,G_3)$, the integral of $\widehat{G}$ with respect to $\widehat{\pi}^N$, also denoted by $\langle\widehat{\pi}^N,\widehat{G}\rangle$, is given by

$$
\langle\widehat{\pi}^N,\widehat{G}\rangle = \sum_{i=1}^{3}\langle\pi^{N,i}\,,\,G_i\rangle\,.
$$

Denote respectively by Δ_N and Δ the discrete Laplacian and the Laplacian defined for any functions $G \in \mathcal{C}^2(\Lambda;\mathbb{R})$ (resp. $G \in \mathcal{C}^2(\mathbb{R}^d;\mathbb{R})$), if $x, x \pm e_j \in \Lambda_N$ (resp. $\mathbb{Z}^d$) for $1 \leqslant j \leqslant d$ and $u \in \Lambda\backslash\Gamma$ (resp. $\mathbb{Z}^d$). by

$$\Delta_N G(x/N) = N^2 \sum_{j=1}^{d} \Big[G((x+e_j)/N) + G((x-e_j)/N) - 2G(x/N) \Big],$$

$$\Delta G(u) = \sum_{j=1}^{d} \partial_{e_j}^2 G(u).$$

We have now all the material to state our results.

5.2.2 Specific entropy and Dirichlet form

Denote by $\Lambda_N^\infty = \{-N, \cdots, N\} \times \mathbb{Z}^{d-1}$, the macroscopic space is $\Lambda^\infty = (-1,1) \times \mathbb{R}^{d-1}$ and its boundary is $\Gamma^\infty := \{(x_1, ..., x_d) \in \Lambda^\infty : x_1 = \pm 1\}$. In this subsection we consider the sub-lattice $\Lambda_{N,n} = \{-N, ..., N\} \times \{-n, ..., n\}^{d-1}$ of Λ_N^∞, for fixed $n \geqslant 1$. Define $\widehat{\Sigma}_{N,n} = (\{0,1\} \times \{0,1\})^{\Lambda_{N,n}}$. We start by defining the two main ingredients needed in the proof of hydrodynamic limit in infinite box : the specific entropy and the specific Dirichlet form of a measure on $\widehat{\Sigma}_N$ with respect to some reference product measure $\nu^N_{\widehat{\theta}(\cdot)}$. For each positive integer n and a measure μ on $\widehat{\Sigma}_N$, we denote by μ_n the marginal of μ on $\widehat{\Sigma}_{N,n}$: For each $(\alpha, \zeta) \in \widehat{\Sigma}_{N,n}$,

$$\mu_n(\alpha,\zeta) = \mu\Big\{(\xi,\omega) \ : \ (\xi(x),\omega(x)) = (\alpha(x),\zeta(x)) \ \text{ for } x \in \Lambda_{N,n}\Big\}\,. \tag{5.2.15}$$

We fix as *reference measure* a product measure $\nu^N_{\widehat{\theta}} := \nu^N_{\widehat{\theta}(\cdot)}$, where $\widehat{\theta} = (\theta_1, \theta_2, \theta_3) : \Lambda^\infty \to (0,1)^3$ is a smooth function with the only requirement that $\widehat{\theta}(\cdot)\big|_{\Gamma^\infty} = \widehat{b}(\cdot)$.

In other words (recall (5.2.6), (5.2.8)), introducing the function $\theta_0(.) = 1 - \theta_1(.) - \theta_2(.) - \theta_3(.)$, we have

$$\begin{aligned} d\nu^N_{\widehat{\theta}(\cdot),n}(\xi,\omega) &= \widehat{Z}^{-1}_{\widehat{\theta},n} \exp\left\{ \sum_{i=1}^{3} \sum_{x \in \Lambda_{N,n}} \left(\log \frac{\theta_i(x/N)}{\theta_0(x/N)} \right) \eta_i(x) \right\} \\ \text{with} \qquad \widehat{Z}^{-1}_{\widehat{\theta},n} &= \prod_{x \in \Lambda_{N,n}} \theta_0(x/N). \end{aligned} \tag{5.2.16}$$

To do changes of variables (detailed in Appendix 5.A), it is convenient to write (5.2.16) as follows :

$$d\nu^N_{\widehat{\theta}(\cdot),n}(\xi,\omega) = \exp\Big\{ \sum_{j=0}^{3} \sum_{x \in \Lambda_{N,n}} \vartheta_j(x/N)\eta_j(x) \Big\} \tag{5.2.17}$$

$$\text{with} \qquad \vartheta_j(x/N) = \log \theta_j(x/N). \tag{5.2.18}$$

For a positive integer n, we denote by $s_n(\mu_n|\nu^N_{\widehat{\theta},n})$ the relative entropy of μ_n with respect to $\nu^N_{\widehat{\theta},n}$ defined by

$$s_n(\mu_n|\nu^N_{\widehat{\theta},n}) = \sup_{U\in C_b(\widehat{\Sigma}_{N,n})} \Big\{ \int U(\xi,\omega)d\mu_n(\xi,\omega) - \log \int e^{U(\xi,\omega)} d\nu^N_{\widehat{\theta},n}(\eta,\xi)\Big\}. \tag{5.2.19}$$

In this formula $C_b(\widehat{\Sigma}_{N,n})$ stands for the space of all bounded continuous functions on $\widehat{\Sigma}_{N,n}$. Since the measure $\nu^N_{\widehat{\theta},n}$ gives a positive probability to each configuration, all the measures on $\widehat{\Sigma}_{N,n}$ are absolutely continuous with respect to $\nu^N_{\widehat{\theta},n}$ and we have an explicit formula for the entropy :

$$s_n(\mu_n|\nu^N_{\widehat{\theta},n}) = \int \log\left(f_n(\xi,\omega)\right) d\mu_n(\xi,\omega), \tag{5.2.20}$$

where f_n is the probability density of μ_n with respect to $\nu^N_{\widehat{\theta},n}$.

Define the Dirichlet form $\mathcal{D}_n(\mu_n|\nu_{\widehat{\theta},n})$ of the measure μ_n with respect to $\nu_{\widehat{\theta},n}$ in the box $\Lambda_{N,n}$

$$\mathcal{D}_n(\mu_n|\nu_{\widehat{\theta},n}) = -\int \sqrt{f_n}(\xi,\omega)\big(\mathfrak{L}_{N,n}\sqrt{f_n}\big)(\xi,\omega)d\nu_{\widehat{\theta},n}(\xi,\omega)\,,$$

where $\mathcal{D}_n(\mu_n|\nu_{\widehat{\theta},n})$ is the restriction of the process to the box $\Lambda_{N,n}$

Let $\mathfrak{L}_{N,n}$ denote the restriction of the generator $\mathfrak{L}_N$ to the box $\Lambda_{N,n}$:

$$\mathfrak{L}_{N,n} = N^2\mathcal{L}_{N,n} + \mathbb{L}_{N,n} + N^2 L_{\widehat{b},N,n}\,, \tag{5.2.21}$$

with

$$\mathcal{L}_{N,n} = \sum_{\substack{x,y\in\Lambda_{N,n}\\ \|x-y\|=1}} \mathcal{L}^{x,y}_N\,, \quad \mathbb{L}_{N,n} = \sum_{x\in\Lambda_{N,n}} \mathbb{L}^x_{N,n}\,, \quad L_{\widehat{b},N,n} = \sum_{x\in\Lambda_{N,n}\cap\Gamma_N} L^x_N\,. \tag{5.2.22}$$

Here for a bond $(x,y)\in\Lambda^2_{N,n}$, $\mathcal{L}^{x,y}_N$ stands for the piece of generator associated to the exchange of particles between the two sites x and y, $\mathbb{L}^x_{N,n}$ corresponds to the flips at site $x\in\Lambda_{N,n}$ for the generalized contact process restricted to $\Lambda_{N,n}$, and for $x\in\Gamma_N$, L^x_N stands for the flips at site x due to the boundary dynamics. We have for $x\in\Lambda_{N,n}$,

$$\begin{aligned}\mathbb{L}^x_{N,n}f(\xi,\omega) \;\; = \;\; & \Big(r(1-\omega(x))+\omega(x)\Big)\Big[f(\xi,\sigma^x\omega)-f(\xi,\omega)\Big] \\ & +\Big(\beta_{N,n}(x,\xi,\omega)\big(1-\xi(x)\big)+\xi(x)\Big)\Big[f(\sigma^x\xi,\omega)-f(\xi,\omega)\Big]\end{aligned} \tag{5.2.23}$$

where

$$\beta_{N,n}(x,\xi,\omega) = \lambda_1 \sum_{\substack{y\in\Lambda_{N,n}\\ \|y-x\|=1}} \xi(y)(1-\omega(y)) + \lambda_2 \sum_{\substack{y\in\Lambda_{N,n}\\ \|y-x\|=1}} \xi(y)\omega(y)\,. \tag{5.2.24}$$

Similarly, we define the corresponding Dirichlet forms,

$$D_n(\mu_n|\nu_{\widehat{\theta},n}) = \mathcal{D}_n^0(\mu_n|\nu_{\widehat{\theta},n}) \;+\; D_n^{\widehat{b}}(\mu_n|\nu_{\widehat{\theta},n})\,,$$

with

$$\begin{aligned}
\mathcal{D}_n^0(\mu_n|\nu_{\widehat{\theta},n}) &= \sum_{\substack{x,y\in\Lambda_{N,n}\\ \|x-y\|=1}} (\mathcal{D}_n^0)^{x,y}(\mu_n|\nu_{\widehat{\theta},n}) \\
D_n^{\widehat{b}}(\mu_n|\nu_{\widehat{\theta},n}) &= \sum_{x\in\Lambda_{N,n}\cap\Gamma_N} (D_n^{\widehat{b}})^{x}(\mu_n|\nu_{\widehat{\theta},n})\,,
\end{aligned}$$

where

$$\begin{aligned}
(\mathcal{D}_n^0)^{x,y}(\mu_n|\nu_{\widehat{\theta},n}) &= \int \Big(\sqrt{f_n(\xi^{x,y},\omega^{x,y})} - \sqrt{f_n(\xi,\omega)}\Big)^2 d\nu_{\widehat{\theta},n}(\xi,\omega),\\
(D_n^{\widehat{b}})^{x}(\mu_n|\nu_{\widehat{\theta},n}) &= \int c_x\big(\widehat{b}(x/N),\xi,\sigma^x\omega\big)\Big(\sqrt{f_n(\xi,\sigma^x\omega)} - \sqrt{f_n(\xi,\omega)}\Big)^2 d\nu_{\widehat{\theta},n}(\xi,\omega)\\
&\quad + \int c_x\big(\widehat{b}(x/N),\sigma^x\xi,\omega\big)\Big(\sqrt{f_n(\sigma^x\xi,\omega)} - \sqrt{f_n(\xi,\omega)}\Big)^2 d\nu_{\widehat{\theta},n}(\xi,\omega)\\
&\quad + \int c_x\big(\widehat{b}(x/N),\sigma^x\xi,\sigma^x\omega\big)\Big(\sqrt{f_n(\sigma^x\xi,\sigma^x\omega)} - \sqrt{f_n(\xi,\omega)}\Big)^2 d\nu_{\widehat{\theta},n}(\xi,\omega)\,.
\end{aligned}$$

We will also need

$$\mathbb{D}_n(\mu_n|\nu_{\widehat{\theta},n}) = \sum_{x\in\Lambda_{N,n}} (\mathbb{D}_n)^x(\mu_n|\nu_{\widehat{\theta},n}) \tag{5.2.25}$$

where

$$\begin{aligned}
(\mathbb{D}_n)^x(\mu_n|\nu_{\widehat{\theta},n}) &= \int \Big(r(1-\omega(x))+\omega(x)\Big)\Big(\sqrt{f_n(\xi,\sigma^x\omega)} - \sqrt{f_n(\xi,\omega)}\Big)^2 d\nu_{\widehat{\theta},n}(\xi,\omega)\\
&\quad + \int \beta_{N,n}(x,\xi,\omega)(1-\xi(x))+\xi(x)\Big)\Big(\sqrt{f_n(\sigma^x\xi,\omega)} - \sqrt{f_n(\xi,\omega)}\Big)^2 d\nu_{\widehat{\theta},n}(\xi,\omega),
\end{aligned}$$

Define the *specific entropy* $\mathcal{S}(\mu|\nu_{\widehat{\theta}}^N)$ and the *Dirichlet form* $\mathfrak{D}(\mu|\nu_{\widehat{\theta}}^N)$ of a measure μ on $\widehat{\Sigma}_N$ with respect to $\nu_{\widehat{\theta}}^N$ as

$$\mathcal{S}(\mu|\nu_{\widehat{\theta}}^N) = N^{-1}\sum_{n\geqslant 1} s_n(\mu_n|\nu_{\widehat{\theta},n}^N)e^{-n/N}, \tag{5.2.26}$$

$$\mathfrak{D}(\mu|\nu_{\widehat{\theta}}^N) = N^{-1}\sum_{n\geqslant 1} D_n(\mu_n|\nu_{\widehat{\theta},n}^N)e^{-n/N}. \tag{5.2.27}$$

Notice that by the entropy convexity and since $\sup_{x\in\Lambda_N}\{\xi(x)+\omega(x)\}$ is finite, for any positive measure μ on $\widehat{\Sigma}_N$ and any integer n, we have

$$s_n(\mu_n|\nu_{\widehat{\theta},n}^N) \;\leqslant\; C_0 N n^{d-1}\,, \tag{5.2.28}$$

for some constant C_0 that depends on $\widehat{\theta}, \lambda_1, \lambda_2, r$ (see Chapter 4 Appendix 4.A). Moreover, by (5.2.26) and (5.2.28), there exists a positive constant $C_0' \equiv C(\widehat{\theta}, \lambda_1, \lambda_2, r)$ such that for any positive measure μ on $\widehat{\Sigma}_N$,

$$\mathcal{S}(\mu|\nu^N_{\widehat{\theta},n}) \leqslant C_0' N^d . \tag{5.2.29}$$

We need more notation. We denote by $(S_N^{\widehat{b}}(t))_{t\geqslant 0}$ the semigroup associated to the generator $\mathfrak{L}_N$. For a measure μ on $\widehat{\Sigma}_N$ we shall denote by $\mu(t)$ the time evolution of the measure μ under the semigroup $S_N^{\widehat{b}} : \mu(t) = \mu S_N^{\widehat{b}}(t)$.

We first prove uniform upper bounds on the entropy production and the Dirichlet form.

Theorem 5.2.1. *Let $\widehat{\theta} : \overline{\Lambda^\infty} \to (0,1)^3$ be a smooth function such that $\widehat{\theta}(\cdot)\big|_{\Gamma^\infty} = \widehat{b}(\cdot)$. For any time $t \geqslant 0$, there exists a positive finite constant $C_1 \equiv C(t, \widehat{\theta}, \lambda_1, \lambda_2, r)$, so that*

$$\int_0^t \mathfrak{D}(\mu(s)|\nu^N_{\widehat{\theta}})\, ds \leqslant C_1 N^{d-2} .$$

To get this result, one needs to consider our system in large finite volume and bound the entropy production in terms of the Dirichlet form. This is given by the following lemma.

Lemma 5.2.1.

$$\partial_t \mathcal{S}(\mu(t)|\nu^N_{\widehat{\theta}}) \leqslant -A_0 N^2 \mathfrak{D}(\mu(t)|\nu^N_{\widehat{\theta}}) + A_1 N^d \tag{5.2.30}$$

5.2.3 Hydrodynamics in a bounded domain.

Suppose in this subsection that $\Lambda_N = \{-N, \cdots, N\} \times \mathbb{T}_N^{d-1}$, the macroscopic space is $\Lambda = (-1,1) \times \mathbb{T}^{d-1}$. Fix $T > 0$. We shall prove in Theorem 5.2.2 below that the macroscopic evolution of the local particles density $\widehat{\pi}^N$ is described by the following system of non-linear reaction-diffusion equations

$$\begin{cases} \partial_t \widehat{\rho} &= \Delta \widehat{\rho} + \widehat{F}(\widehat{\rho}) \quad \text{in} \quad \Lambda \times (0,T), \\ \widehat{\rho}_0(\cdot) &= \widehat{\gamma}(\cdot) \quad \text{in} \quad \Lambda, \\ \widehat{\rho}_t|_\Gamma &= \widehat{b}(\cdot) \quad \text{for} \;\; 0 \leqslant t \leqslant T , \end{cases} \tag{5.2.31}$$

where $\widehat{F} = (F_1, F_2, F_3) : [0,1] \to \mathbb{R}^3$ is given by

$$\begin{cases} F_1(\rho_1,\rho_2,\rho_3) &= 2d(\lambda_1\rho_1 + \lambda_2\rho_3)\rho_0 + \rho_3 - \rho_1(r+1) , \\ F_2(\rho_1,\rho_2,\rho_3) &= r\rho_0 + \rho_3 - 2d(\lambda_1\rho_1 + \lambda_2\rho_3)\rho_2 - \rho_2 , \\ F_3(\rho_1,\rho_2,\rho_3) &= 2d(\lambda_1\rho_1 + \lambda_2\rho_3)\rho_2 + r\rho_1 - 2\rho_3 . \end{cases} \tag{5.2.32}$$

where $\rho_0 = 1 - \rho_1 - \rho_2 - \rho_3$. By weak solution of (5.2.31) we mean a function $\widehat{\rho}(\cdot,\cdot) : [0,T] \times \Lambda \to \mathbb{R}^3$ satisfying

(B1) For any $i \in \{1,2,3\}$, $\rho_i \in L^2\left((0,T); H^1(\Lambda)\right)$:

$$\sum_{i=1}^{3} \int_0^T ds \Big(\int_\Lambda \| \nabla \rho_i(s,u) \|^2 du \Big) < \infty \,.$$

(B2) For every function $\widehat{G}(t,u) = \widehat{G}_t(u) = (G_{1,t}(u), G_{2,t}(u), G_{3,t}(u))$ in $\mathcal{C}_0^{1,2}\big([0,T] \times \overline{\Lambda}; \mathbb{R}^3\big)$, we have

$$\begin{aligned} &\langle \widehat{\rho}_T(\cdot), \widehat{G}_T(\cdot) \rangle - \langle \widehat{\rho}_0(\cdot), \widehat{G}_0(\cdot) \rangle - \int_0^T ds \, \langle \widehat{\rho}_s(\cdot), \partial_s \widehat{G}_s(\cdot) \rangle \\ &= \int_0^T ds \, \langle \widehat{\rho}_s(\cdot), \Delta \widehat{G}_s(\cdot) \rangle + \int_0^T ds \, \langle \widehat{F}(\rho_s)(\cdot), \widehat{G}_s(\cdot) \rangle \\ &\quad - \sum_{i=1}^{3} \int_0^T ds \int_\Gamma \mathbf{n}_1(r) \, b_i(r) (\partial_1 G_{i,s})(r) \, dS(r) \,, \end{aligned} \tag{5.2.33}$$

where $\mathcal{C}_0^{1,2}\big([0,T] \times \Lambda; \mathbb{R}^3\big)$ is the space of functions from $[0,T] \times \Lambda$ to $\mathbb{R}^3$ twice continuously differentiable in Λ with continuous time derivative and vanishing at the boundary Γ of Λ. Here $\mathbf{n}=(\mathbf{n}_1, \ldots, \mathbf{n}_d)$ stands for the outward unit normal vector to the boundary surface Γ and dS for an element of surface on Γ. For $G, H \in L^2(\Lambda)$, $\langle G(\cdot), H(\cdot) \rangle$ is the usual scalar product of $L^2(\Lambda)$:

$$\langle G(\cdot), H(\cdot) \rangle = \sum_{i=1}^{3} \int_\Lambda G_i(u) H_i(u) du$$

(B3) $\widehat{\rho}(0,u) = \widehat{\gamma}(u)$ a.e.

Let $\mathcal{M}_+^1$ be the subset of $\mathcal{M}$ of all positive measures absolutely continuous with respect to the Lebesgue measure with positive density bounded by 1 :

$$\mathcal{M}_+^1 = \big\{ \pi \in \mathcal{M} : \pi(du) = \rho(u) du \quad \text{and} \quad 0 \leqslant \rho(u) \leqslant 1 \;\text{ a.e.} \big\} \,.$$

Let $D([0,T], (\mathcal{M}_+^1)^3)$ be the set of right continuous with left limits trajectories with values in $(\mathcal{M}_+^1)^3$, endowed with the Skorohod topology and equipped with its Borel $\sigma-$ algebra. For a probability measure μ on $\widehat{\Sigma}_N$ denote by $(\xi_t, \omega_t)_{t \in [0,T]}$ the Markov process with generator $\mathfrak{L}_N$ with initial distribution μ. Denote by $\mathbb{P}_\mu^{N,\widehat{b}}$ the probability measure on the path space $D([0,T], \widehat{\Sigma}_N)$ corresponding to the Markov process $(\xi_t, \omega_t)_{t \in [0,T]}$ and by $\mathbb{E}_\mu^{N,\widehat{b}}$ the expectation with respect to $\mathbb{P}_\mu^{N,\widehat{b}}$. We denote by $\widehat{\pi}^N$ the map from $D([0,T], \widehat{\Sigma}_N)$ to $D([0,T], (\mathcal{M}_+^1)^3)$ defined by $\widehat{\pi}^N(\xi_., \omega_.)_t = \widehat{\pi}^N(\xi_t, \omega_t)$ and by $Q_\mu^{N,\widehat{b}} = \mathbb{P}_\mu^{N,\widehat{b}} \circ (\widehat{\pi}^N)^{-1}$ the law of the process $\big(\widehat{\pi}^N(\xi_t, \omega_t)\big)_{t \in [0,T]}$.

We shall prove :

Theorem 5.2.2. *Let $(\mu_N)_{N\geqslant 1}$ be a sequence of initial probability measures such that μ_N is a probability measure on $\widehat{\Sigma}_N$ for each N. The sequence of probability measures $(Q^{N,\hat{b}}_{\mu_N})_{N\geqslant 1}$ is weakly relatively compact and all its converging subsequences converge to some limit $Q^{\hat{b},*}$ that is concentrated on absolutely continuous paths whose densities $\widehat{\rho}\in\mathcal{C}([0,T],(\mathcal{M}^1_+)^3)$ satisfy (B1) and (B2).*

Moreover, if for any $\delta>0$ and for any function $\widehat{G}\in\mathcal{C}^0(\overline{\Lambda};\mathbb{R}^3)$

$$\lim_{N\to\infty}\mu_N\Big\{\Big|\langle\widehat{\pi}_N(\xi,\omega),\widehat{G}(\cdot)\rangle-\langle\widehat{\gamma}(\cdot),\widehat{G}(\cdot)\rangle\Big|\geqslant\delta\Big\}=0\,,\tag{5.2.34}$$

for an initial continuous profile $\widehat{\gamma}:\Lambda\to[0,1]^3$, then the sequence of probability measures $(Q^{N,\hat{b}}_{\mu_N})_{N\geqslant 1}$ converges to the Dirac measure concentrated on the unique weak solution $\widehat{\rho}(\cdot,\cdot)$ of the boundary value problem (5.2.31). Accordingly, for any $t\in[0,T]$, any $\delta>0$ and any function $\widehat{G}\in\mathcal{C}^{1,2}\big([0,T]\times\overline{\Lambda};\mathbb{R}^3\big)$

$$\lim_{N\to\infty}\mathbb{P}^{N,\hat{b}}_{\mu_N}\Big\{\Big|\langle\widehat{\pi}_N(\xi_t,\omega_t),\widehat{G}(\cdot)\rangle-\langle\widehat{\rho}_t(\cdot),\widehat{G}(\cdot)\rangle\Big|\geqslant\delta\Big\}=0\,.$$

We will prove Theorem 5.2.2 in Section 5.4.

5.2.4 Currents.

In this subsection, we will study the evolution of the empirical currents, namely the conservative current (cf. [3]) and the non-conservative current one (cf. [9]).

For $t\geqslant 0$, $1\leqslant i\leqslant 3$, $1\leqslant j\leqslant d$ such that $x,x+e_j\in\Lambda_N$, denote by $J^{x,x+e_j}_t(\eta_i)$ the total number of particles of type i that jumped from x to $x+e_j$ before time t and by $W^{x,x+e_j}_t(\eta_i)=J^{x,x+e_j}_t(\eta_i)-J^{x+e_j,x}_t(\eta_i)$ the conservative current of particles of type i across the bond $\{x,x+e_j\}$ before time t. The corresponding conservative empirical measure $\mathbb{W}^N_t$ is the product finite signed measure on Λ_N defined as $\mathbb{W}^N_t(\eta_i)=(W^N_{1,t}(\eta_i),\ldots,W^N_{d,t}(\eta_i))\in\mathcal{M}_d=\{\mathcal{M}(\Lambda)\}^d$, where for $1\leqslant j\leqslant d$, $1\leqslant i\leqslant 3$,

$$W^N_{j,t}(\eta_i)\;=\;N^{-(d+1)}\sum_{x,x+e_j\in\Lambda_N}W^{x,x+e_j}_t(\eta_i)\delta_{x/N}\,.$$

For a continuous vector field $\mathbf{G}=(G_1,\ldots,G_d)\in\mathcal{C}_c(\Lambda;\mathbb{R}^d)$ the integral of $\mathbf{G}$ with respect to $\mathbb{W}^N_t(\eta_i)$, also denoted by $\langle\mathbb{W}^N_t(\eta_i),\mathbf{G}\rangle$, is given by

$$\langle\mathbb{W}^N_t(\eta_i),\mathbf{G}\rangle\;=\;\sum_{j=1}^d\langle W^N_{j,t}(\eta_i),G_j\rangle\,.\tag{5.2.35}$$

Finally, we introduce the signed measure $\widehat{\mathbb{W}^N_t}(\widehat{\eta})=(\mathbb{W}^N_t(\eta_1),\mathbb{W}^N_t(\eta_2),\mathbb{W}^N_t(\eta_3))\in(\mathcal{M}_d)^3$ and for $\widehat{\mathbf{G}}=(\mathbf{G}_1,\mathbf{G}_2,\mathbf{G}_3)\in(\mathcal{C}_c(\Lambda;\mathbb{R}^d))^3$ the notation

$$\langle\widehat{\mathbb{W}^N_t},\widehat{\mathbf{G}}\rangle=\sum_{i=1}^3\langle\mathbb{W}^N_t(\eta_i),\mathbf{G}_i\rangle\,.$$

For $x \in \Lambda_N$, we denote by $Q_t^x(\eta_i)$ the total number of particles of type i created minus the total number of particles of type i annihilated at site x before time t. The corresponding non-conservative empirical measure is

$$Q_t^N(\eta_i) = \frac{1}{N^d} \sum_{x \in \Lambda_N} Q_t^x(\eta_i) \delta_{x/N}.$$

We introduce the signed measure $\widehat{Q}_t^N = (Q_t^N(\eta_1), Q_t^N(\eta_2), Q_t^N(\eta_3)) \in \mathcal{M}_3$ and for $\widehat{H} = (H_1, H_2, H_3) \in (\mathcal{C}_c(\Lambda; \mathbb{R}))^3$ the notation

$$\langle \widehat{Q}_t^N, \widehat{H} \rangle = \sum_{i=1}^{3} \langle Q_t^N(\eta_i), H_i \rangle.$$

We can now state the law of large numbers for the current :

Proposition 5.2.1. *Fix a smooth initial profile $\widehat{\gamma} : \Lambda \to \mathbb{R}^3$. Let (μ_N) be a sequence of probability measures satisfying* (5.2.34) *and $\widehat{\rho}$ be the weak solution of the system of equations* (5.2.31)*. Then, for each $T > 0$, $\delta > 0$, $\widehat{\mathbf{G}} \in \big(\mathcal{C}_c^1(\Lambda; \mathbb{R}^d)\big)^3$ and $\widehat{H} \in \big(\mathcal{C}_c^1(\Lambda; \mathbb{R})\big)^3$,*

$$\lim_{N \to \infty} \mathbb{P}_{\mu_N}^{N,\hat{b}} \Big[\big\langle \widehat{\mathbb{W}_T^N}, \widehat{\mathbf{G}} \big\rangle \; - \; \int_0^T dt \big\langle \big\{ - \nabla \widehat{\rho}_t \big\} \, , \, \widehat{\mathbf{G}} \big\rangle \Big| > \delta \Big] \; = \; 0 \, , \tag{5.2.36}$$

$$\lim_{N \to \infty} \mathbb{P}_{\mu_N}^{N,\hat{b}} \Big[\big\langle \widehat{Q}_T^N, \widehat{H} \big\rangle \; - \; \int_0^T dt \big\langle \widehat{F}(\widehat{\rho}_t) \, , \, \widehat{H} \big\rangle \Big| > \delta \Big] \; = \; 0 \, . \tag{5.2.37}$$

We shall prove Proposition 5.2.1 in Section 5.5.

5.2.5 Hydrodynamics in infinite volume.

In this subsection we derive the hydrodynamic limit to the generalized contact process in $\mathbb{Z}^d$ with generator $\mathfrak{L}$ defined in (5.2.14). For a fixed density profile $\widehat{\theta}$, denote by $\nu_{\widehat{\theta}}$ the product measure on $\widehat{\Sigma}$ such that $\theta_i = \mathbb{E}_{\nu_{\widehat{\theta}}}\big[\eta_i(0)\big]$.

Theorem 5.2.3. *Consider a sequence of probability measures $(\mu_N)_{N \geqslant 1}$ on $\widehat{\Sigma}$ associated to a continuous profile $\widehat{\gamma} : \mathbb{R}^d \to [0,1]^3$, that is, for all continuous function $\widehat{G} \in \mathcal{C}_c(\mathbb{R}^d; \mathbb{R}^3)$,*

$$\lim_{N \to \infty} \mu_N \Big(\big| \frac{1}{N^d} \sum_i \sum_{x \in \mathbb{Z}^d} G(x/N) \eta_i(x) - \langle \widehat{\gamma}, \widehat{G} \rangle \big| > \delta \Big) = 0$$

for all $\delta > 0$. Then for all $t \geqslant 0$,

$$\lim_{N \to \infty} P_{\mu_N}^N \Big(\Big| \frac{1}{N^d} \sum_i \sum_{x \in \mathbb{Z}^d} G_t(x/N) \eta_{i,t}(x) - \langle \widehat{\rho}_t(\cdot), \widehat{G}(\cdot) \rangle \Big| \geqslant \delta \Big) = 0$$

for any function $\widehat{G} \in \mathcal{C}_c(\mathbb{R}^d;\mathbb{R}^3)$ and $\delta > 0$, where $\widehat{\rho}(t,u)$ is the unique weak solution of the system

$$\begin{cases} \partial_t \widehat{\rho} = \Delta \widehat{\rho} + \widehat{F}(\widehat{\rho}) \quad in \quad \mathbb{Z}^d \times (0,T), \\ \widehat{\rho}_0(\cdot) = \widehat{\gamma}(\cdot) \quad in \quad \mathbb{Z}^d, \end{cases} \tag{5.2.38}$$

A weak solution $\widehat{\rho}(\cdot,\cdot)$ of (5.2.38) satisfies the following assertions :

(IV1) For any $i \in \{1,2,3\}$, $\rho_i \in L^\infty\big([0,T]\big) \times \mathbb{R}^d\big)$.

(IV2) For every function $\widehat{G}(t,u) = \widehat{G}_t(u) = (G_{1,t}(u), G_{2,t}(u), G_{3,t}(u))$ in $\mathcal{C}_c^{1,2}\big([0,T] \times \mathbb{R}^d;\mathbb{R}^3\big)$, we have

$$\begin{aligned} &\langle \widehat{\rho}_T(\cdot), \widehat{G}_T(\cdot)\rangle - \langle \widehat{\rho}_0(\cdot), \widehat{G}_0(\cdot)\rangle - \int_0^T ds \, \langle \widehat{\rho}_s(\cdot), \partial_s \widehat{G}_s(\cdot)\rangle \\ &\quad = \int_0^T ds \, \langle \widehat{\rho}_s(\cdot), \Delta \widehat{G}_s(\cdot)\rangle + \int_0^T ds \, \langle \widehat{F}(\rho_s)(\cdot), \widehat{G}_s(\cdot)\rangle \end{aligned} \tag{5.2.39}$$

(IV3) $\widehat{\rho}(0,u) = \widehat{\gamma}(u)$ a.e.

We shall prove Theorem 5.2.3 in Section 5.6.

Remark 5.2.1. *As a consequence of Theorem 5.2.3, the law of large numbers for the currents stated in Proposition 5.2.1 still holds in infinite volume, since the corresponding proof given in Section 5.5 only relies on the hydrodynamic limit.*

5.2.6 Uniqueness of weak solutions

In this subsection, we state the results concerning the uniqueness of the weak solution to the equations of the boundary driven case in finite volume case and in infinite volume. Begin with the finite volume case :

Lemma 5.2.2 (Uniqueness in finite volume)**.** *For any $T > 0$, the system* (5.2.31) *has a unique weak solution in the class* $\big(L^2\big([0,T];H^1(\Lambda)\big)^3$.

Fix $T > 0$. Let $\widehat{\gamma} = (\gamma_1,\gamma_2,\gamma_3) : \Lambda^\infty \to [0,1]^3$ be a smooth initial profile, and denote by $\widehat{\rho} = (\rho_1,\rho_2,\rho_3) : [0,T] \times \Lambda^\infty \to [0,1]^3$ a typical macroscopic trajectory. The macroscopic evolution of the local particles density $\widehat{\pi}^N$ over Λ_N^∞ is described by the system of the following non-linear reaction-diffusion equations with Dirichlet boundary conditions :

$$\begin{cases} \partial_t \widehat{\rho} &= \Delta \widehat{\rho} + \widehat{F}(\widehat{\rho}) \quad \text{in} \quad \Lambda^\infty \times (0,T), \\ \widehat{\rho}_0(\cdot) &= \widehat{\gamma}(\cdot) \quad \text{in} \quad \Lambda^\infty, \\ \widehat{\rho}_t|_\Gamma &= \widehat{b}(\cdot) \quad \text{for} \ \ 0 \leqslant t \leqslant T, \end{cases} \tag{5.2.40}$$

where $\widehat{F} = (F_1,F_2,F_3) : [0,1] \to \mathbb{R}^3$ was given in (5.2.32). By weak solution of (5.2.40) we mean a function $\widehat{\rho}(\cdot,\cdot) : [0,T] \times \Lambda^\infty \to \mathbb{R}^3$ satisfying

(IB1) For any $i \in \{1,2,3\}$, $\rho_i \in L^\infty\left((0,T) \times \Lambda^\infty\right)$.

(IB2) For every function $\widehat{G}(t,u) = \widehat{G}_t(u) = (G_{1,t}(u), G_{2,t}(u), G_{3,t}(u))$ in $\mathcal{C}_0^{1,2}\big([0,T]\times \Lambda^\infty;\mathbb{R}^3\big)$, we have

$$\begin{aligned}
&\langle \widehat{\rho}_T(\cdot), \widehat{G}_T(\cdot)\rangle - \langle \widehat{\rho}_0(\cdot), \widehat{G}_0(\cdot)\rangle - \int_0^T ds\, \langle \widehat{\rho}_s(\cdot), \partial_s \widehat{G}_s(\cdot)\rangle \\
&= \int_0^T ds\, \langle \widehat{\rho}_s(\cdot), \Delta \widehat{G}_s(\cdot)\rangle + \int_0^T ds\, \langle \widehat{F}(\rho_s)(\cdot), \widehat{G}_s(\cdot)\rangle \\
&\quad - \sum_{i=1}^3 \int_0^T ds \int_{\Gamma^\infty} \mathbf{n}_1(r)\, b_i(r) (\partial_1 G_{i,s})(r)\, dS(r)\,,
\end{aligned} \tag{5.2.41}$$

(IB3) $\widehat{\rho}(0,u) = \widehat{\gamma}(u)$. a.e.

We now state the following proposition :

Proposition 5.2.2 (Uniqueness in infinite volume with stochastic reservoirs)**.** *For any $T > 0$, the system of equations* (5.2.40) *has a unique weak solution in the class* $\big(L^\infty\big([0,T]\times\Lambda^\infty\big)\big)^3$.

We prove these results in Section 5.7.

5.3 Proof of the specific entropy (Theorem 5.2.1)

In this section we prove Theorem 5.2.1 and Lemma 5.2.1.

Proof of Theorem 5.2.1. Integrate the expression (5.2.30) from 0 to t and use 5.2.29. □

Proof of Lemma 5.2.1. For a measure μ_n on $\widehat{\Sigma}_{N,n}$, denote by f_n^t the density of $\mu_n(t)$ with respect to $\nu^N_{\widehat{\theta},n}$. For any subset $A \subset \Lambda$ and any function $f \in L^1(\nu^N_{\widehat{\theta}})$, denote by $\langle f\rangle_A$ the function on $(\{0,1\}\times\{0,1\})^{\Lambda\setminus A}$ obtained by integrating f with respect to $\nu^N_{\widehat{\theta}}$ over the coordinates $\{(\xi(x),\omega(x)), x \in A\}$. In the case where $A = \Lambda_{N,n+1}\setminus\Lambda_{N,n}$, we simplify the expectation by $\langle f\rangle_{n+1}$. Following the Kolmogorov forward equation, one has

$$\partial_t f_n^t = \langle \mathfrak{L}^*_{N,n+1} f^t_{n+1}\rangle_{n+1}, \tag{5.3.1}$$

where $\mathfrak{L}^*_{N,n}$ stands for the adjoint operator of $\mathfrak{L}_{N,n}$ in $L^2(\nu^N_{\widehat{\theta},n})$. From the convexity of the entropy (5.2.28) and by (5.3.1),

$$\begin{aligned}
\partial_t s_n(\mu_n|\nu^N_{\widehat{\theta},n}) &= \partial_t \int f_n^t \log f_n^t d\nu^N_{\widehat{\theta}(\cdot),n} = \int \log f_n^t \mathfrak{L}^*_{N,n+1} f^t_{n+1} d\nu^N_{\widehat{\theta},n+1} \\
&= N^2 \int \log f_n^t \mathcal{L}^*_{N,n+1} f^t_{n+1} d\nu^N_{\widehat{\theta},n+1} + \int \log f_n^t \mathbb{L}^*_{N,n+1} f^t_{n+1} d\nu^N_{\widehat{\theta},n+1} \\
&\quad + N^2 \int \log f_n^t L^*_{N,n+1} f^t_{n+1} d\nu^N_{\widehat{\theta},n+1}\,.
\end{aligned} \tag{5.3.2}$$

Denote the last three integrals by Ω_1, Ω_2 and Ω_3 respectively. Recall that $\nu^N_{\widehat{\theta},n}$ stands for the measure associated to a smooth profile $\widehat{\theta} : \Lambda^\infty \to (0,1)^3$ such that $\widehat{\theta}\big|^{\infty}_{\Gamma} = \widehat{b}(\cdot)$. We now derive bounds on Ω_1, Ω_2 and Ω_3.

Bound on Ω_1. We shall decompose the generator $\mathcal{L}_{N,n+1}$ into a part associated to exchanges within $\Lambda_{N,n}$ and a part associated to exchanges at the boundaries, that is, denoting $\Lambda^c_{N,n} = \Lambda_N \backslash \Lambda_{N,n}$,

$$\begin{aligned}
\Omega_1 &= N^2 \int f^t_{n+1} \mathcal{L}_{N,n+1}(\log f^t_n) d\nu^N_{\widehat{\theta},n+1} \\
&= N^2 \sum_{\substack{(x,y)\in\Lambda_{N,n}\times\Lambda_{N,n}\\ \|x-y\|=1}} \int f^t_{n+1} \mathcal{L}^{x,y}_N (\log f^t_n) d\nu^N_{\widehat{\theta},n+1} \\
&\quad + N^2 \sum_{\substack{(x,y)\in\Lambda_{N,n}\times\Lambda^c_{N,n}\\ \|x-y\|=1}} \int f^t_{n+1} \mathcal{L}^{x,y}_N (\log f^t_n) d\nu^N_{\widehat{\theta},n+1} \\
&= N^2 \sum_{\substack{(x,y)\in\Lambda_{N,n}\times\Lambda_{N,n}\\ \|x-y\|=1}} \Omega^{(1)}_1(x,y) && (5.3.3) \\
&\quad + N^2 \sum_{\substack{(x,y)\in\Lambda_{N,n}\times\Lambda^c_{N,n}\\ \|x-y\|=1}} \Omega^{(2)}_1(x,y)\,. && (5.3.4)
\end{aligned}$$

Successively, for the term (5.3.3),

$$\begin{aligned}
\Omega^{(1)}_1(x,y) &= \int f^t_{n+1}(\xi,\omega)\Big(\log f^t_n(\xi^{x,y},\omega^{x,y}) - \log f^t_n(\xi,\omega)\Big) d\nu^N_{\widehat{\theta},n+1}(\xi,\omega) \\
&= \int \langle f^t_{n+1}(\xi,\omega)\rangle_{n+1} \log \frac{f^t_n(\xi^{x,y},\omega^{x,y})}{f^t_n(\xi,\omega)} d\nu^N_{\widehat{\theta},n}(\xi,\omega) \\
&\leqslant -\int \Big(\sqrt{f^t_n(\xi^{x,y},\omega^{x,y})} - \sqrt{f^t_n(\xi,\omega)}\Big)^2 d\nu^N_{\widehat{\theta},n}(\xi,\omega) \\
&\quad + \int \Big(f^t_n(\xi^{x,y},\omega^{x,y}) - f^t_n(\xi,\omega)\Big) d\nu^N_{\widehat{\theta},n}(\xi,\omega), \\
&= -(\mathcal{D}^0_n)^{x,y}(\mu_n|\nu_{\widehat{\theta},n}) + \int \mathcal{L}^{x,y}_N f^t_n(\xi,\omega) d\nu^N_{\widehat{\theta},n}(\xi,\omega) && (5.3.5)
\end{aligned}$$

where we used the fact that for any $a,b>0$,

$$a(\log b - \log a) \leqslant -\big(\sqrt{b}-\sqrt{a}\big)^2 + (b-a). \qquad (5.3.6)$$

Now, by a change of variables $(\alpha,\beta) = (\xi^{x,y},\omega^{x,y})$, using Lemma 5.A.1 we have

$$\int \mathcal{L}^{x,y}_N f^t_n(\xi,\omega) d\nu^N_{\widehat{\theta},n}(\xi,\omega) = \sum_{0\leqslant i\neq j\leqslant 3} \int \eta_j(y)\eta_i(x) R^{x,y}_{i,j}(\widehat{\theta}) f^t_n(\xi,\omega) d\nu^N_{\widehat{\theta},n}(\xi,\omega) \qquad (5.3.7)$$

where

$$R_{i,j}^{x,y}(\widehat{\theta}) = \exp\Big((\vartheta_j(y/N) - \vartheta_j(x/N)) - (\vartheta_i(y/N) - \vartheta_i(x/N)) \Big) - 1\,. \tag{5.3.8}$$

By a Taylor expansion, (5.3.7) is of order $O(N^{-1})$.

For the part (5.3.4) associated to the boundaries, we shall write for each pair $(x, y) \in \Lambda_{N,n} \times \Lambda_{N,n}^c$ with $\|x - y\| = 1$,

$$\mathcal{L}_N^{x,y} = \sum_{0 \leqslant i \neq j \leqslant 3} \mathcal{L}_{i \leftrightarrow j}^{x,y} \tag{5.3.9}$$

where $\mathcal{L}_{i \leftrightarrow j}^{x,y}$ stands for the exchange of values i and j at the boundaries.

$$\begin{aligned} \mathcal{L}_{i \leftrightarrow j}^{x,y} f(\xi, \omega) &= \eta_i(x)\eta_j(y)\Big(f(\xi^{x,y}, \omega^{x,y}) - f(\xi, \omega) \Big) \\ &\quad + \eta_j(x)\eta_i(y)\Big(f(\xi^{x,y}, \omega^{x,y}) - f(\xi, \omega) \Big)\,. \end{aligned} \tag{5.3.10}$$

So that,

$$\begin{aligned} \Omega_1^{(2)}(x,y) \quad &= \quad \sum_{0 \leqslant i \neq j \leqslant 3} \int f_{n+1}^t \mathcal{L}_{i \leftrightarrow j}^{x,y} \log f_n^t(\xi,\omega) d\nu_{\widehat{\theta},n+1}^N(\xi,\omega) \\ &= \quad \sum_{0 \leqslant i \neq j \leqslant 3} \int \eta_i(x)\eta_j(y) f_{n+1}^t(\xi,\omega) \log \frac{f_n^t(\xi^{x,y}, \omega^{x,y})}{f_n^t(\xi,\omega)} d\nu_{\widehat{\theta},n+1}^N(\xi,\omega) \\ &\quad + \sum_{0 \leqslant i \neq j \leqslant 3} \int \eta_j(x)\eta_i(y) f_{n+1}^t(\xi,\omega) \log \frac{f_n^t(\xi^{x,y}, \omega^{x,y})}{f_n^t(\xi,\omega)} d\nu_{\widehat{\theta},n+1}^N(\xi,\omega)\,. \end{aligned} \tag{5.3.11}$$

Let us detail the computation for $i = 1$ and $j = 3$, the other values would be deduced in a similar way. In this case, by a change of variables $(\xi', \omega') = (\xi^{x,y}, \omega^{x,y})$ in the integral corresponding to $i = 1, j = 3$ in the second term of the r.h.s. (5.3.11) using Lemma 5.A.1, and noticing for the integral corresponding to $i = 1, j = 3$ in the first term of the r.h.s. (5.3.11) that $\xi^{x,y} = \xi$ since $i = 1, j = 3$, we have

$$\begin{aligned} &\int f_{n+1}^t \mathcal{L}_{1 \leftrightarrow 3}^{x,y} \log f_n^t(\xi,\omega) d\nu_{\widehat{\theta},n+1}^N(\xi,\omega) \\ &\quad = \int \eta_1(x)\eta_3(y) f_{n+1}^t(\xi,\omega) \log \frac{f_n^t(\xi, \omega^{x,y})}{f_n^t(\xi,\omega)} d\nu_{\widehat{\theta},n+1}^N(\xi,\omega) \\ &\qquad + \int \eta_1(x)\eta_3(y) \exp\Big((\vartheta_3(y/N) - \vartheta_3(x/N)) - (\vartheta_1(y/N) - \vartheta_1(x/N)) \Big) \\ &\qquad\qquad\qquad \times f_{n+1}^t(\xi^{x,y}, \omega^{x,y}) \log \frac{f(\xi,\omega)}{f(\xi^{x,y}, \omega^{x,y})} d\nu_{\widehat{\theta},n+1}^N(\xi,\omega) \end{aligned}$$

$$
\begin{aligned}
&= \int \eta_1(x)\langle \eta_3(y) f^t_{n+1}(\xi,\omega)\rangle_{n+1} \log \frac{f^t_n(\xi,\sigma^x\omega)}{f^t_n(\xi,\omega)} d\nu^N_{\widehat{\theta},n}(\xi,\omega) \\
&\quad + \int R^{x,y}_{1,3}(\widehat{\theta})\eta_1(x)\langle \eta_3(y) f^t_{n+1}(\xi,\omega^{x,y})\rangle_{n+1} \log \frac{f(\xi,\omega)}{f(\xi,\sigma^x\omega)} d\nu^N_{\widehat{\theta},n}(\xi,\omega) \\
&\quad + \int \eta_1(x)\langle \eta_3(y) f^t_{n+1}(\xi,\omega^{x,y})\rangle_{n+1} \log \frac{f(\xi,\omega)}{f(\xi,\sigma^x\omega)} d\nu^N_{\widehat{\theta},n}(\xi,\omega)\,,
\end{aligned}
$$

where $R^{x,y}_{i,j}(\widehat{\theta})$ was defined in (5.3.8). By a Taylor expansion of $R^{x,y}_{i,j}(\widehat{\theta})$, the second line on the last r.h.s. is of order $O(N^{-1})$. We deduce that

$$
\begin{aligned}
&\int f^t_{n+1}\mathcal{L}^{x,y}_{1\leftrightarrow 3} \log f^t_n(\xi,\omega) d\nu^N_{\widehat{\theta},n+1}(\xi,\omega) \\
&= \int \eta_1(x)\Big(\langle F^{(1)}_{1,3}(\xi,\omega)\rangle_{n+1} - \langle F^{(2)}_{1,3}(\xi,\omega)\rangle_{n+1}\Big) \log \frac{f^t_n(\xi,\sigma^x\omega)}{f^t_n(\xi,\omega)} d\nu^N_{\widehat{\theta},n}(\xi,\omega) + O(N^{-1})
\end{aligned}
\tag{5.3.12}
$$

where

$$
F^{(1)}_{i,j}(\xi,\omega) = \eta_j(y) f^t_{n+1}(\xi,\omega)\,, \quad F^{(2)}_{i,j}(\xi,\omega) = \eta_j(y) f^t_{n+1}(\xi^{x,y},\omega^{x,y})\,. \tag{5.3.13}
$$

If we now define

$$
\begin{aligned}
E_1(i,j) &= \{(\xi,\omega) : \langle F^{(1)}_{i,j}(\xi,\omega)\rangle_{n+1} \geqslant \langle F^{(2)}_{i,j}(\xi,\omega)\rangle_{n+1}, \\
&\qquad f^t_n(\xi,\sigma^x\omega) \geqslant f^t_n(\xi,\omega)\} && (5.3.14)\\
E_2(i,j) &= \{(\xi,\omega) : \langle F^{(1)}_{i,j}(\xi,\omega)\rangle_{n+1} \leqslant \langle F^{(2)}_{i,j}(\xi,\omega)\rangle_{n+1}, \\
&\qquad f^t_n(\xi,\sigma^x\omega) \leqslant f^t_n(\xi,\omega)\} && (5.3.15)
\end{aligned}
$$

the integral in the r.h.s. of (5.3.12) is non-negative on $E_1(1,3)\cup E_2(1,3)$. Then, thanks to the inequalities (we shall make a high use of them)

$$
\begin{aligned}
\log a &\leqslant 2(\sqrt{a}-1) && (5.3.16)\\
2ab &\leqslant \frac{N}{A}a^2 + \frac{A}{N}b^2 && (5.3.17)
\end{aligned}
$$

for any positive a, b, A, the integral in the r.h.s. of (5.3.12) is bounded by

$$
\begin{aligned}
&\int_{E_1(1,3)\cup E_2(1,3)} \eta_1(x)\Big(\langle F^{(1)}_{1,3}(\xi,\omega)\rangle_{n+1} - \langle F^{(2)}_{1,3}(\xi,\omega)\rangle_{n+1}\Big) \log \frac{f^t_n(\xi,\sigma^x\omega)}{f^t_n(\xi,\omega)} d\nu^N_{\widehat{\theta},n}(\xi,\omega) \\
&\quad \leqslant 2\int_{E_1(1,3)\cup E_2(1,3)} \eta_1(x)\Big(\langle F^{(1)}_{1,3}(\xi,\omega)\rangle_{n+1} - \langle F^{(2)}_{1,3}(\xi,\omega)\rangle_{n+1}\Big) \\
&\qquad\qquad \times \Big(\sqrt{\frac{f^t_n(\xi,\sigma^x\omega)}{f^t_n(\xi,\omega)}} - 1\Big) d\nu^N_{\widehat{\theta},n}(\xi,\omega)
\end{aligned}
$$

$$\leqslant \frac{N}{A}\int_{E_1(1,3)\cup E_2(1,3)} \eta_1(x)\Big(\sqrt{\langle F^{(1)}_{1,3}(\xi,\omega)\rangle_{n+1}} - \sqrt{\langle F^{(2)}_{1,3}(\xi,\omega)\rangle_{n+1}}\Big)^2 d\nu^N_{\hat\theta,n}(\xi,\omega)$$
$$+ \frac{A}{N}\int_{E_1(1,3)\cup E_2(1,3)} \Big(\sqrt{\langle F^{(1)}_{1,3}(\xi,\omega)\rangle_{n+1}} + \sqrt{\langle F^{(2)}_{1,3}(\xi,\omega)\rangle_{n+1}}\Big)^2$$
$$\times\Big(\sqrt{\frac{f^t_n(\xi,\sigma^x\omega)}{f^t_n(\xi,\omega)}} - 1\Big)^2 d\nu^N_{\hat\theta,n}(\xi,\omega)\,. \tag{5.3.18}$$

To bound the first integral of the last r.h.s. in (5.3.18) by a piece of Dirichlet form, apply Cauchy-Schwarz inequality so that

$$\frac{N}{A}\int_{E_1(1,3)\cup E_2(1,3)} \eta_1(x)\Big(\sqrt{\langle F^{(1)}_{1,3}(\xi,\omega)\rangle_{n+1}} - \sqrt{\langle F^{(2)}_{1,3}(\xi,\omega)\rangle_{n+1}}\Big)^2 d\nu^N_{\hat\theta,n}(\xi,\omega)$$
$$= \frac{N}{A}\frac{1}{N}\sum_{m=n+1}^{n+N}\int_{E_1(1,3)\cup E_2(1,3)} \eta_1(x)\Big(\sqrt{\langle \eta_3(y) f^t_m(\xi^{x,y},\omega^{x,y})\rangle_{\Lambda_{N,m}\setminus\Lambda_{N,n}}}$$
$$- \sqrt{\langle \eta_3(y) f^t_m(\xi,\omega)\rangle_{\Lambda_{N,m}\setminus\Lambda_{N,n}}}\Big)^2 d\nu^N_{\hat\theta,m}(\xi,\omega)$$
$$\leqslant \frac{1}{A}\sum_{m=n+1}^{n+N}\int_{E_1(1,3)\cup E_2(1,3)} \eta_1(x)\Big\langle \eta_3(y)\Big(\sqrt{f^t_m(\xi^{x,y},\omega^{x,y})}$$
$$- \sqrt{f^t_m(\xi,\omega)}\Big)^2\Big\rangle_{\Lambda_{N,m}\setminus\Lambda_{N,n}} d\nu^N_{\hat\theta,m}(\xi,\omega)$$
$$= \frac{1}{A}\sum_{m=n+1}^{n+N}\int_{E_1(1,3)\cup E_2(1,3)} \eta_1(x)\eta_3(y)\Big(\sqrt{f^t_m(\xi^{x,y},\omega^{x,y})} - \sqrt{f^t_m(\xi,\omega)}\Big)^2 d\nu^N_{\hat\theta,m}(\xi,\omega) \tag{5.3.19}$$
$$\leqslant \frac{1}{A}\sum_{m=n+1}^{n+N}\int_{E_1(1,3)\cup E_2(1,3)} \Big(\sqrt{f^t_m(\xi^{x,y},\omega^{x,y})} - \sqrt{f^t_m(\xi,\omega)}\Big)^2 d\nu^N_{\hat\theta,m}(\xi,\omega) \tag{5.3.20}$$

which is of order $O(N)$. Now, to bound the second integral of the last r.h.s. in (5.3.18), we separate the integrations on $E_1(1,3)$ and on $E_2(1,3)$. We first look at the integral on $E_1(1,3)$, to get

$$\frac{A}{N}\int_{E_1(1,3)} \eta_1(x)\Big(\sqrt{\langle F^{(1)}_{1,3}(\xi,\omega)\rangle_{n+1}} + \sqrt{\langle F^{(2)}_{1,3}(\xi,\omega)\rangle_{n+1}}\Big)^2$$
$$\times\Big(\sqrt{\frac{f^t_n(\xi,\sigma^x\omega)}{f^t_n(\xi,\omega)}} - 1\Big)^2 d\nu^N_{\hat\theta,n}(\xi,\omega)$$
$$\leqslant \frac{4A}{N}\int_{E_1(1,3)} \eta_1(x)\frac{\langle F^{(1)}_{1,3}(\xi,\omega)\rangle_{n+1}}{f^t_n(\xi,\omega)}\Big(\sqrt{f^t_n(\xi,\sigma^x\omega)} - \sqrt{f^t_n(\xi,\omega)}\Big)^2 d\nu^N_{\hat\theta,n}(\xi,\omega)$$
$$\leqslant \frac{4A}{N}\int_{E_1(1,3)} \eta_1(x)\Big(f^t_n(\xi,\sigma^x\omega) - 2\sqrt{f^t_n(\xi,\sigma^x\omega)}\sqrt{f^t_n(\xi,\omega)} + f^t_n(\xi,\omega)\Big) d\nu^N_{\hat\theta,n}(\xi,\omega)$$

$$\begin{aligned}
&\leqslant \frac{4A}{N}\int_{E_1(1,3)}\eta_1(x)\Big(f_n^t(\xi,\sigma^x\omega)-f_n^t(\xi,\omega)\Big)d\nu^N_{\hat\theta,n}(\xi,\omega)\\
&\leqslant \frac{4A}{N}\int \eta_1(x)f_n^t(\xi,\sigma^x\omega)d\nu^N_{\hat\theta,n}(\xi,\omega)=\int \eta_3(x)e^{(\vartheta_1(x/N)-\vartheta_3(x/N))}f_n^t(\xi,\omega)d\nu^N_{\hat\theta,n}(\xi,\omega)\\
&\leqslant \frac{AC_1}{N}
\end{aligned}\tag{5.3.21}$$

for some positive constant C_1. We have used the definition (5.3.14) of $E_1(1,3)$ for the first and third inequalities, the definition (5.3.13) of $F^{(1)}_{1,3}(\xi,\omega)$ with the bound $\langle F^{(1)}_{1,3}(\xi,\omega)\rangle_{n+1}\leqslant\langle f^t_{n+1}(\xi,\omega)\rangle_{n+1}=f^t_n(\xi,\omega)$ for the second inequality, Lemma 5.A.2*(iii)* for the equality, (5.2.18), (5.2.11) and that f_n^t is a probability density to conclude.

We now look at the integral on $E_2(1,3)$, to get

$$\begin{aligned}
&\frac{A}{N}\int_{E_2(1,3)}\eta_1(x)\Big(\sqrt{\langle F^{(1)}_{1,3}(\xi,\omega)\rangle_{n+1}}+\sqrt{\langle F^{(2)}_{1,3}(\xi,\omega)\rangle_{n+1}}\Big)^2\\
&\qquad\qquad\qquad\times\Big(\sqrt{\frac{f_n^t(\xi,\sigma^x\omega)}{f_n^t(\xi,\omega)}}-1\Big)^2d\nu^N_{\hat\theta,n}(\xi,\omega)\\
&\leqslant \frac{4A}{N}\int_{E_2(1,3)}\eta_1(x)\frac{\langle F^{(2)}_{1,3}(\xi,\omega)\rangle_{n+1}}{f_n^t(\xi,\omega)}\Big(\sqrt{f_n^t(\xi,\sigma^x\omega)}-\sqrt{f_n^t(\xi,\omega)}\Big)^2d\nu^N_{\hat\theta,n}(\xi,\omega)\\
&\leqslant \frac{8A}{N}\int_{E_2(1,3)}\eta_1(x)\frac{\langle F^{(2)}_{1,3}(\xi,\omega)\rangle_{n+1}}{f_n^t(\xi,\omega)}f_n^t(\xi,\omega)d\nu^N_{\hat\theta,n}(\xi,\omega)\\
&\leqslant \frac{8A}{N}\int \eta_3(x)\eta_1(y)e^{(\vartheta_3(y/N)-\vartheta_3(x/N))-(\vartheta_1(y/N)-\vartheta_1(x/N))}f^t_{n+1}(\xi,\omega)d\nu^N_{\hat\theta,n+1}(\xi,\omega)\\
&\leqslant \frac{AC_1'}{N}
\end{aligned}\tag{5.3.22}$$

for some positive constant C_1'. We have used the definition (5.3.15) of $E_2(1,3)$ for the first and second inequalities, the definition (5.3.13) of $F^{(2)}_{1,3}(\xi,\omega)$ with Lemma 5.A.1 for the third inequality, and (5.2.18), (5.2.11) and finally that f_n^t is a probability density.

To conclude to an upper bound of Ω_1, combining (5.3.5) with (5.3.20), (5.3.21), (5.3.22)

$$\Omega_1\leqslant -N^2\mathcal{D}^0_n(\mu_n(t)|\nu^N_{\hat\theta,n})+C_1''ANn^{d-1}\tag{5.3.23}$$

Bound on Ω_2. We decompose the generator of the reaction part into a part involving only sites within $\Lambda_{N,n}$ and a part involving sites in $\Lambda_{N,n+1}\backslash\Lambda_{N,n}$. Recalling (5.2.21), (5.2.22), we have

$$\Omega_2 \;=\; \int f^t_{n+1}\mathbb{L}_{N,n+1}\log f_n^t d\nu^N_{\hat\theta,n+1}=\int f^t_{n+1}\mathbb{L}_{N,n}\log f_n^t d\nu^N_{\hat\theta,n+1}+\Omega_2^{(1)}$$

Proceeding as in (5.3.5), we get

$$\int f_{n+1}^t \mathbb{L}_{N,n} \log f_n^t d\nu_{\hat{\theta},n+1}^N \leqslant -\mathbb{D}_n(\mu_n(t)|\nu_{\hat{\theta},n}) + \int \mathbb{L}_{N,n} f_n^t d\nu_{\hat{\theta},n}^N \tag{5.3.24}$$

The second term on the r.h.s. is of order $O(Nn^{d-1})$ since the rates $\beta_{N,n}(.,.)$ are bounded. And, denoting $\partial\Lambda_{N,n} = \{x \in \Lambda_{N,n} : \exists y \in \Lambda_{N,n}^c, \|y - x\| = 1\}$,

$$\begin{aligned} \Omega_2^{(1)} &= \sum_{x \in \partial\Lambda_{N,n}} \int f_{n+1}^t(\xi,\omega)\Big(\lambda_1 \sum_{\substack{y \in \Lambda_{N,n}^c \\ \|y-x\|=1}} \xi(y)(1-\omega(y)) \\ &\quad + \lambda_2 \sum_{\substack{y \in \Lambda_{N,n}^c \\ \|y-x\|=1}} \xi(y)\omega(y)\Big)(1-\xi(x)) \log \frac{f_n^t(\sigma^x\xi,\omega)}{f_n^t(\xi,\omega)} d\nu_{\hat{\theta},n+1}^N(\xi,\omega) \end{aligned}$$

which is of order $O(Nn^{d-1})$ in an analogous way to the computation done for $\Omega_1^{(2)}$, using inequalities (5.3.16)–(5.3.17). Combined with (5.3.24)–(5.3.24), one has

$$\Omega_2 \leqslant -\mathbb{D}_n(\mu_n(t)|\nu_{\hat{\theta},n}^N) + K_2 N n^{d-1} \tag{5.3.25}$$

Bound on Ω_3. Since $L_{\hat{b},N,n} = \sum_{x \in \Lambda_{N,n} \cap \Gamma_N} L_N^x$, using inequality (5.3.6) we have Since $\nu_{\hat{\theta},n}^N$ is reversible with respect to the generator $L_{\hat{b},N,n}$, using inequality (5.3.6),

$$\begin{aligned} \Omega_3 &= N^2 \sum_{x \in \Lambda_{N,n} \cap \Gamma_N} \int f_{n+1}^t L_N^x \log f_n^t d\nu_{\hat{\theta},n+1}^N \\ &= N^2 \sum_{x \in \Lambda_{N,n} \cap \Gamma_N} \int \langle f_{n+1}^t(\xi,\omega)\rangle_{n+1} L_N^x \log f_n^t d\nu_{\hat{\theta},n}^N \\ &\leqslant -N^2 D_n^{\hat{b}}(\mu_n(t)|\nu_{\hat{\theta},n}^N) + N^2 \sum_{x \in \Lambda_{N,n} \cap \Gamma_N} \int L_N^x f_n^t d\nu_{\hat{\theta},n}^N \\ &= -N^2 D_n^{\hat{b}}(\mu_n(t)|\nu_{\hat{\theta},n}^N) \end{aligned} \tag{5.3.26}$$

It is for the last equality that we needed $\nu_{\hat{\theta},n}^N$ to be reversible with respect to the generator $L_{\hat{b},N,n}$.

The estimate (5.3.26), together with (5.3.23) and (5.3.25), gives us

$$\partial_t s_n(\mu_n(t)|\nu_{\hat{\theta},n}^N) \leqslant -N^2 \mathcal{D}_n^0(\mu_n(t)|\nu_{\hat{\theta}(\cdot),n}^N) + (K_2 + C_1'' A) N n^{d-1} - N^2 D_n^{\hat{b}}(\mu_n(t)|\nu_{\hat{\theta},n}^N)$$

Therefore, multiplying by $\exp(-n/N)$ and summing over $n \in \mathbb{N}$, one gets (5.2.30) with $A_0 = 1$ and $A_1 = (K_2 + AC_1'')$. □

5.4 Hydrodynamics in a bounded domain

We now turn into the proof of the hydrodynamics in bounded domain. Denote by $\nu^N_{\hat\theta(\cdot)}$, the reference measure restricted to Λ_N. Let us consider, for any function $\widehat{G} \in (C_0^2([0,T]\times\overline{\Lambda};\mathbb{R}))^3$,

$$M_t^{N,i}(\widehat{G}) = \langle \pi_t^{N,i}, G_{i,t}\rangle - \langle \pi_0^N, G_{i,0}\rangle - \int_0^t \langle \pi_s^{N,i}, \partial_s G_{i,s}\rangle ds - \int_0^t \mathfrak{L}_N \langle \pi_s^{N,i}, G_{i,s}\rangle ds \quad (5.4.1)$$

which is a martingale for all $i = 1, 2, 3$. Our goal will be to make this martingale converge, and for this, first we compute :

$$\begin{aligned} N^2 \mathcal{L}_N \langle \pi_t^{N,i}, G_i\rangle = \langle \pi_t^{N,i}, \Delta_N G_i\rangle - \frac{1}{N^{d-1}} \sum_{x\in\Gamma_N^+} \partial_{e_1}^N G_i((x-e_1)/N)\eta_i(x) \\ + \frac{1}{N^{d-1}} \sum_{x\in\Gamma_N^-} \partial_{e_1}^N G_i(x/N)\eta_i(x) \quad (5.4.2) \end{aligned}$$

where $\Gamma_N^{\pm} = \{(u_1,\ldots,u_d) \in \overline{\Lambda}_N : u_1 = \pm N\}$ amd $\partial_{e_1}^N$ stands for the discrete gradient : $\partial_{e_1}^N G(x/N) = N\Big(G((x+e_1)/N) - G(x/N)\Big)$, with $x, x+e_1 \in \Lambda_N$, as well as

$$\mathbb{L}_N\eta_1(0) = \beta_N(0,\xi,\omega)\eta_0(0) + \eta_3(0) - (r+1)\eta_1(0), \quad (5.4.3)$$
$$\mathbb{L}_N\eta_2(0) = r\eta_0(0) + \eta_3(0) - \beta_N(0,\xi,\omega)\eta_2(0) - \eta_2(0), \quad (5.4.4)$$
$$\mathbb{L}_N\eta_3(0) = \beta_N(0,\xi,\omega)\eta_2(0) + r\eta_1(0) - 2\eta_3(0), \quad (5.4.5)$$

Note that since $\widehat{G}$ vanishes at the boundaries on $\overline{\Lambda}$, $L_{\hat b,N}\langle \pi_t^{N,i}, G_i\rangle = 0$. To get to the system of equations (5.2.31), we shall need to replace the local functions appearing in (5.4.3)–(5.4.5) by functions of the empirical measures, thanks to the replacement lemma.

5.4.1 Replacement lemma.

One main step in proving the hydrodynamic limit of a system lies in being able to replace local functions by functions of the density fields to close the equations. For any $\ell \in \mathbb{N}$, define the empirical mean densities in a box of size $(2\ell+1)^d$ centred at x by $\widehat{\eta}^\ell(x) = (\eta_1^\ell(x), \eta_2^\ell(x), \eta_3^\ell(x))$:

$$\eta_i^\ell(x) = \frac{1}{(2\ell+1)^d} \sum_{\|y-x\|\leqslant \ell} \eta_i(y), \text{ for all } i = 1,2,3.$$

For any cylinder function ϕ, $\widetilde{\phi}(\widehat{\theta})$ stands for the expectation of $\phi(\xi,\omega)$ with respect to $\nu^N_{\hat\theta}$, so that we can define for any $\epsilon > 0$,

$$V_{\epsilon N}(\xi,\omega) = \Big| \frac{1}{(2\epsilon N+1)^d} \sum_{\|y\|\leqslant \epsilon N} \tau_y \phi(\xi,\omega) - \widetilde{\phi}(\widehat{\eta}^{\epsilon N}(0)) \Big|, \quad (5.4.6)$$

where $\widehat{\eta}^k(x) = (\eta_1^k, \eta_2^k, \eta_3^k)(x)$.

Lemma 5.4.1 (replacement lemma)**.** *For any $G \in \mathcal{C}^\infty([0,T] \times \overline{\Lambda}, \mathbb{R})$ and any $\widehat{H} \in \mathcal{C}^\infty([0,T] \times \overline{\Lambda}; \mathbb{R}^3)$,*

$$\overline{\lim_{\epsilon \to 0}}\, \overline{\lim_{N \to \infty}}\, \mathbb{P}^{N,\hat{b}}_{\mu_N}\Big(\frac{1}{N^d} \sum_{x \in \Lambda_N} \int_0^T |G_s(x/N)| \tau_x V_{\epsilon N}(\xi_s, \omega_s) ds \geqslant \delta\Big) = 0, \tag{5.4.7}$$

for any $\delta > 0$, and

$$\overline{\lim_{N \to \infty}}\, \mathbb{E}^{N,\hat{b}}_{\mu_N}\Big(\Big|\int_0^T \frac{1}{N^{d-1}} \sum_{x \in \Gamma_N} \partial^N_{e_1} H_t(x/N) \mathbf{n}_1(x/N) \Big(\eta_{i,t}(x) - b_i(x/N)\Big) ds\Big|\Big) = 0. \tag{5.4.8}$$

for all $i = 1, 2, 3$.

Before proving the replacement lemma, let us state the so-called one and two blocks estimates. The one block estimate ensures the average of local functions in some large microscopic boxes can be replaced by their mean with respect to the grand-canonical measure parametrized by the particles density in these boxes. While the two blocks estimate ensures the particles density over large microscopic boxes are close to the one over small macroscopic boxes :

Lemma 5.4.2 (One block estimate)**.** *Fix a constant profile $\widehat{\rho} = (\rho_1, \rho_2, \rho_3) \in (0,1)^3$,*

$$\overline{\lim_{k \to \infty}}\, \overline{\lim_{N \to \infty}} \sup_{f : \mathcal{D}^0_N(f) \leqslant C N^{d-2}} \int \frac{1}{N^d} \sum_{x \in \Lambda_N} \tau_x V_k(\xi, \omega) f(\xi, \omega) d\nu^N_{\widehat{\rho}, N}(\xi, \omega) = 0 \tag{5.4.9}$$

where for $k \in \mathbb{N}$, $V_k(\xi, \omega)$ was defined in (5.4.6)*.*

Lemma 5.4.3 (Two blocks estimate)**.** *Given a constant profile $\widehat{\rho} = (\rho_1, \rho_2, \rho_3) \in (0,1)^3$, for all $i = 1, 2, 3$,*

$$\overline{\lim_{k \to \infty}}\, \overline{\lim_{\epsilon \to 0}}\, \overline{\lim_{N \to \infty}} \sup_{f : \mathcal{D}^0_N(f) \leqslant C N^{d-2}} \sup_{|h| \leqslant \epsilon N} \frac{1}{N^d} \int \sum_{x \in \Lambda_N} \big|\eta^k_i(x+h) - \eta^{\epsilon N}_i(x)\big| f(\xi, \omega) d\nu^N_{\widehat{\rho}}(\xi, \omega) = 0. \tag{5.4.10}$$

Here, $\mathcal{D}^N_0$ denotes the Dirichlet form associated to the generator of exchanges of particles in Λ_N.

Proof of Proposition 5.4.1. First deal with the proof of (5.4.7). By Markov's inequality,

$$\mathbb{P}^{N,\hat{b}}_{\mu_N}\Big(\frac{1}{N^d} \sum_{x \in \Lambda_N} \int_0^T |G_s(x/N)| \tau_x V_{\epsilon N}(\xi_s, \omega_s) ds \geqslant \delta\Big) \leqslant \frac{1}{\delta} \|G\|_\infty \mathbb{E}^{N,\hat{b}}_{\mu_N}\Big(\frac{1}{N^d} \sum_{x \in \Lambda_N} \int_0^T \tau_x V_{\epsilon N}(\xi_s, \omega_s) ds\Big)$$

Let $a > 0$ be decreasing to zero after ϵ, and a smooth function $\widehat{\theta}_a = (\theta_{a,1}, \theta_{a,2}, \theta_{a,3}) : \Lambda \to (0,1)^3$, equal in $\Lambda_{(1-a)N} = [-1+a, 1-a] \times \mathbb{T}_N^{d-1}$ to some constant, say $\widehat{\alpha}$, and to $\widehat{b}$ at the boundaries. As $\sup_{k,(\xi,\omega),x} \tau_x V_k(\xi,\omega) < \infty$, we have

$$\frac{1}{N^d} \sum_{x \in \Lambda_N \setminus \Lambda_{(1-a)N}} \int_0^T \tau_x V_{\epsilon N}(\xi_s, \omega_s) ds \leqslant aTC_0,$$

for some positive constant C_0. Therefore,

$$\mathbb{E}_{\mu_N}^{N,\hat{b}}\Big(\frac{1}{N^d} \sum_{x \in \Lambda_N} \int_0^T \tau_x V_{\epsilon N}(\xi_s, \omega_s) ds\Big) \leqslant aTC_0 + \mathbb{E}_{\mu_N}^{N,\hat{b}}\left(\frac{1}{N^d} \sum_{x \in \Lambda_{(1-a)N}} \int_0^T \tau_x V_{\epsilon N}(\xi_s, \omega_s) ds\right).$$

Denote by $\bar{f}^T = T^{-1} \displaystyle\int_0^T f_N^s ds$, where f_N^t stands for the density of $\mu_N(t)$ with respect to $\nu_{\hat{\theta}_a}^N$. Since $\Lambda_{(1-a)N}$ is finite, proceeding as in the proof of Theorem 5.2.1 for Ω_1, there exists some positive constant $C_1(a)$ such that the remaining expectation above is bounded by

$$\frac{T}{N^d} \int \sum_{x \in \Lambda_{(1-a)N}} \tau_x V_{\epsilon N}(\xi,\omega) \bar{f}^T(\xi,\omega) d\nu_{\hat{\theta}_a}^N(\xi,\omega) - \gamma T N^{2-d} \mathcal{D}_N^0(\bar{f}^T) + \gamma C_1(a),$$

for all positive γ. Recall $\widehat{\theta}_a$ is equal to some constant $\widehat{\alpha}$ within $\Lambda_{(1-a)N}$. To prove (5.4.7), it thus remains to show that for every positive γ, a,

$$\varlimsup_{\epsilon \to 0} \varlimsup_{N \to \infty} \sup_f \left(\frac{1}{N^d} \int \sum_{x \in \Lambda_{(1-a)N}} \tau_x V_{\epsilon N}(\xi,\omega) f(\xi,\omega) d\nu_{\hat{\alpha}}^N(\xi,\omega) \ - \ \gamma N^{2-d} \mathcal{D}_N^0(f) \right) \ = \ 0,$$

where the supremum is carried over all densities f with respect to $\nu_{\hat{\alpha}}^N$ such that $\mathcal{D}_N^0(f) \leqslant CN^{d-2}$. This result is a consequence of the one and two blocks estimates (5.4.2)–(5.4.3), for which we refer to Chapter 4 since we reduced ourselves to the interior of the domain. Conclude by letting $\gamma \to 0$, then, $a \to 0$.

Now, let us prove the limit (5.4.8). Define

$$W_i^{H_t}(\xi_t, \omega_t)(x) = \partial_{e_1}^N H_t(x/N)\Big(\eta_{i,t}(x) - b_i(x/N)\Big) \tag{5.4.11}$$

Decomposing the outward unit normal vector into both directions,

$$\begin{aligned}
&\mathbb{E}_{\mu_N}^{N,\hat{b}}\Big(\Big|\int_0^T \frac{1}{N^{d-1}} \sum_{x \in \Gamma_N} \partial_{e_1}^N H_{i,s}(x/N) \mathbf{n}_1(x/N)\Big(\eta_{i,s}(x) - b_i(x/N)\Big) ds\Big|\Big). \\
&\quad \leqslant \mathbb{E}_{\mu_N}^{N,\hat{b}}\Big(\Big|\int_0^T \frac{1}{N^{d-1}} \sum_{x \in \Gamma_N^+} W_i^{H_s}(\xi_s, \omega_s)(x) ds\Big|\Big)
\end{aligned}$$

$$+\ \mathbb{E}^{N,\hat{b}}_{\mu_N}\Big(\Big|\int_0^T \frac{1}{N^{d-1}}\sum_{x\in\Gamma_N^-} W_i^{H_s}(\xi_s,\omega_s)(x)ds\Big|\Big),$$

where $\Gamma_N^\pm = \{(u_1,\dots,u_d)\in \pm N\times \mathbb{T}_N^{d-1}\}$. From now, consider the sum over Γ_N^+ as the proof will be similar for the negative part. By the entropy inequality, for any positive γ,

$$\begin{aligned}
&\mathbb{E}^{N,\hat{b}}_{\mu_N}\Big(\Big|\int_0^T \frac{1}{N^d}\sum_{x\in\Gamma_N^+}\partial^N_{e_1}H_{i,s}(x/N)\Big(\eta_{i,s}(x)-b_i(x/N)\Big)ds\Big|\Big).\\
&\quad\leqslant \frac{1}{\gamma N^d}\log \mathbb{E}^{N,\hat{b}}_{\nu^N_{\hat{\theta}}}\left(\exp\Big(\gamma N^d\Big|\int_0^T \frac{1}{N^{d-1}}\sum_{x\in\Gamma_N^+} W_i^{H_s}(\xi_s,\omega_s)(x)ds\Big|\Big)+\frac{s_N(\mu_N|\nu^N_{\hat{\theta}})}{\gamma N^d}\right.
\end{aligned}$$

where $s_N(\mu_N|\nu^N_{\hat{\theta}})$ was defined in (5.2.20). By (5.2.28), there exists some constant K_0 such that $s_N(\mu_N|\nu^N_{\hat{\theta}})\leqslant K_0N^d$. Using that $e^{|a|}\leqslant e^a+e^{-a}$ and

$$\varlimsup_N N^{-d}\log\big(a_N+b_N\big)\leqslant \max\Big(\varlimsup_N N^{-d}\log a_N, \varlimsup_N N^{-d}b_N\Big),$$

one can pull off the absolute value even if it means replacing H by $-H$. By Feynman-Kac formula,

$$\begin{aligned}
&\frac{1}{\gamma N^d}\log \mathbb{E}^{N,\hat{b}}_{\nu^N_{\hat{\theta}}}\left(\exp\Big(\gamma N^d\int_0^T \frac{1}{N^{d-1}}\sum_{x\in\Gamma_N^+}\partial^N_{e_1}H_{i,s}(x/N)\Big(\eta_{i,s}(x)-b_i(x/N)\Big)ds\Big)\right.\\
&\quad\leqslant \left.\int_0^T\sup_f\Big\{\int\frac{1}{N^{d-1}}\sum_{x\in\Gamma_N^+}W_i^{H_s}(\xi_s,\omega_s)(x)f(\xi,\omega)d\nu^N_{\hat{\theta}}(\xi,\omega)+\frac{1}{\gamma N^d}\langle\mathfrak{L}_N\sqrt{f},\sqrt{f}\rangle\Big\}ds\right)
\end{aligned}\tag{5.4.12}$$

where the supremum is carried over all densities f with respect to $\nu^N_{\hat{\rho}}$. By Lemma 5.C.1,

$$\langle\mathfrak{L}_N\sqrt{f},\sqrt{f}\rangle\leqslant -N^2D^{\hat{b}}_N(f)+A_0N^d.\tag{5.4.13}$$

for some positive constant A_0. We now consider the expression $W_i^{H_t}(\xi_s,\omega_s)(x)$ between brackets in (5.4.12) and thanks to changes of variables given in Lemma 5.A.2,

$$\begin{aligned}
&\int\partial^N_{e_1}H_{i,t}(x/N)\Big(\eta_i(x)-b_i(x/N)\Big)f(\xi,\omega)d\nu^N_{\hat{\theta}}(\xi,\omega)\\
&\quad=\int\partial^N_{e_1}H_{i,t}(x/N)\Big(\eta_i(x)\sum_{j\neq i}b_j(x/N)-b_i(x/N)\sum_{j\neq i}\eta_j(x/N)\Big)f(\xi,\omega)d\nu^N_{\hat{\theta}}(\xi,\omega)
\end{aligned}\tag{5.4.14}$$

We detail for instance the case $i=1$, others follow the same way, this is equal to

$$\int\partial^N_{e_1}H_{1,t}(x/N)\Big(\eta_1(x)\Big(b_0(x/N)+b_2(x/N)+b_3(x/N)\Big)$$

$$
\begin{aligned}
&\quad - b_1(x/N)\Big(\eta_0(x) + \eta_2(x) + \eta_3(x)\Big)\Big) f(\xi,\omega) d\nu^N_{\hat{\theta}}(\xi,\omega) \\
&\leqslant \int \partial^N_{e_1} H_{1,t}(x/N)\Bigg(\Big(\eta_1(x) b_0(x/N) - b_1\eta_0(x)\Big) f(\xi,\omega) \\
&\quad + \Big(\eta_1(x) b_3(x/N) - b_1\eta_3(x)\Big) f(\xi,\omega) + b_2(x/N) f(\xi,\omega)\Bigg) d\nu^N_{\hat{\theta}}(\xi,\omega) \\
&= \int \partial^N_{e_1} H_{1,t}(x/N)\Bigg(b_1\eta_0(x)\Big(f(\sigma^x\xi,\omega) - f(\xi,\omega)\Big) \\
&\quad + b_1\eta_3(x)\Big(f(\xi,\sigma^x\omega) 0 f(\xi,\omega)\Big) + b_2(x/N) f(\xi,\omega)\Bigg) d\nu^N_{\hat{\theta}}(\xi,\omega) \\
&\leqslant \int \frac{b_1\eta_0(x) AN}{2}\Big(\sqrt{f(\sigma^x\xi,\omega)} - \sqrt{f(\xi,\omega)}\Big)^2 \\
&\quad + \frac{b_1\eta_3(x) AN}{2}\Big(\sqrt{f(\xi,\sigma^x\omega)} - \sqrt{f(\xi,\omega)}\Big)^2 d\nu^N_{\hat{\theta}}(\xi,\omega) + \frac{C_1'}{2AN}\|\partial^N_{e_1} H_{1,t}(x/N)^2\|_\infty
\end{aligned}
$$

where C_1' is some constant, we used (5.3.17) to get

$$
\begin{aligned}
&\Big(f(\sigma^x\xi,\omega) - f(\xi,\omega)\Big) \\
&\quad = \Big(\sqrt{f(\sigma^x\xi,\omega)} - \sqrt{f(\xi,\omega)}\Big)\Big(\sqrt{f(\sigma^x\xi,\omega)} + \sqrt{f(\xi,\omega)}\Big) \\
&\quad \leqslant \frac{AN}{2}\Big(\sqrt{f(\sigma^x\xi,\omega)} - \sqrt{f(\xi,\omega)}\Big)^2 + \frac{1}{2AN}\Big(\sqrt{f(\sigma^x\xi,\omega)} + \sqrt{f(\xi,\omega)}\Big)^2,
\end{aligned}
$$

and that f is a density while expanding the last term. Overall, dealing with the sum over i, since parts of the Dirichlet form $(D^{\hat{b}}_N)^x$ appear, (5.4.14) is bounded by

$$
\frac{AN}{2}(D^{\hat{b}}_N)^x(f) + \frac{C'}{2AN}\|\partial^N_{e_1} H_{1,t}(x/N)^2\|_\infty
$$

Now summing over $\{x \in \Gamma_N\}$ and multiplying by N^{1-d}, (5.4.12) is bounded by

$$
\Big(\frac{AN^{2-d}}{2} - \frac{N^{2-d}}{\gamma}\Big) D^{\hat{b}}_N(f) + \frac{C}{2AN}\|\partial^N_{e_i} H_{1,t}(x/N)^2\|_\infty + \frac{A_0}{\gamma}
$$

Choose $A = 2/\gamma$ and conclude by letting tend $\gamma \to \infty$, $N \to \infty$. □

5.4.2 Energy estimate.

We now deal with an energy estimate that allows us to exclude paths with infinite energy. For $G \in \mathcal{C}^\infty_c([0,T] \times \Lambda, \mathbb{R})$, define

$$
\widehat{\mathcal{Q}}(\hat{\pi}) = \sup_{i=1,2,3} \sup_{G \in \mathcal{C}^\infty_c([0,T]\times\Lambda,\mathbb{R})} \mathcal{Q}_G(\pi^i) \tag{5.4.15}
$$

where $\mathcal{Q}_G(\pi^i) = \sum_{j=1}^d \int_0^T \int_\Lambda \pi_t^i(u)\partial_{e_j} G_t(u)\, dt\, du - \frac{1}{2}\int_0^T \int_\Lambda G_t(u)^2\, dt\, du.$

Lemma 5.4.4. *Fix a dense sequence $(G_\ell)_{\ell\geqslant 1}$ in $\mathcal{C}_c^\infty([0,T]\times\Lambda,\mathbb{R})$. For all $i=1,2,3$, there exists a constant C_0 such that for any sequence $\{\mu_N : N\geqslant 1\}$ of probability measures on $\widehat{\Sigma}_N$, every $k\geqslant 1$,*

$$\varlimsup_{N\to\infty} \mathbb{E}_{\mu_N}^{N,\hat{b}}\Bigg(\max_{1\leqslant\ell\leqslant k}\Big(\int_0^T \Big(A_i^{G_{\ell,s}}(\xi_s,\omega_s) - \frac{1}{2N^d}\sum_{x\in\Lambda_N} G_{\ell,s}(x/N)^2\Big)ds\Big)\Bigg) \leqslant C_0. \tag{5.4.16}$$

where $A_i^{G_{\ell,t}}(\xi_t,\omega_t) := N^{1-d}\sum_{j=1}^d \sum_{x,x+e_j\in\Lambda_N} (\eta_{i,t}(x+e_j)-\eta_{i,t}(x))G_{\ell,t}(x/N)$.

Proof. By the entropy inequality, for all $\gamma>0$,

$$\begin{aligned}&\mathbb{E}_{\mu_N}^{N,\hat{b}}\Big(\max_{1\leqslant\ell\leqslant k}\int_0^T A_i^{G_{\ell,s}}(\xi_s,\omega_s)ds\Big)\\ &\quad\leqslant \frac{1}{\gamma N^d}\log \mathbb{E}_{\nu_{\hat\theta}^N}^{N,\hat{b}}\Bigg(\exp\Big(\max_{1\leqslant\ell\leqslant k}\Big(\gamma N^d\int_0^T A_i^{G_{\ell,s}}(\xi_s,\omega_s)ds\Big)\Big)\Bigg) + \frac{1}{\gamma N^d}s_N(\mu_N|\nu_{\hat\theta}^N),\end{aligned}$$

where $s_N(\mu_N|\nu_{\hat\theta}^N)$ stands for the relative entropy of μ_N with respect to $\nu_{\hat\theta}^N$ defined in (5.2.20). By (5.2.28), $s_N(\mu_N|\nu_{\hat\theta}^N)\leqslant C_0N^d$, for some constant C_0. Using that

$$\exp\big(\max_{1\leqslant\ell\leqslant k} a_\ell\big) \leqslant \sum_{1\leqslant\ell\leqslant k}\exp(a_\ell)$$

and

$$\varlimsup_N N^{-d}\log\big(\sum_{1\leqslant\ell\leqslant k} a_\ell\big) \leqslant \max_{1\leqslant\ell\leqslant k}\varlimsup_N N^{-d}\log a_\ell,$$

$$\begin{aligned}&\varlimsup_{N\to\infty}\mathbb{E}_{\mu_N}^{N,\hat{b}}\Big(\max_{1\leqslant\ell\leqslant k}\int_0^T A_i^{G_{\ell,s}}(\xi_s,\omega_s)ds\Big)\\ &\quad\leqslant \max_{1\leqslant\ell\leqslant k}\varlimsup_{N\to\infty}\frac{1}{\gamma N^d}\log\mathbb{E}_{\nu_{\hat\theta}^N}^{N,\hat{b}}\Bigg(\exp\Big(\gamma N^d\int_0^T A_i^{G_{\ell,s}}(\xi_s,\omega_s)ds\Big)\Bigg) + \frac{C_0}{\gamma}.\end{aligned}$$

By Feynman-Kac formula,

$$\begin{aligned}&\frac{1}{\gamma N^d}\log\mathbb{E}_{\nu_{\hat\theta}^N}^{N,\hat{b}}\Bigg(\exp\Big(\gamma N^d\int_0^T A_i^{G_{\ell,s}}(\xi_s,\omega_s)ds\Big)\Bigg)\\ &\quad\leqslant \int_0^T \sup_f\Bigg\{\int A_i^{G_{\ell,s}}(\xi_s,\omega_s)f(\xi,\omega)d\nu_{\hat\theta}^N(\xi,\omega) + \frac{1}{\gamma N^d}\langle\mathfrak{L}_N f,f\rangle\Bigg\}ds\end{aligned} \tag{5.4.17}$$

where the supremum is carried over all densities f with respect to $\nu_{\widehat{\theta}}^N$. Writing the supremum over positive densities, it is bounded by

$$\int_0^T \sup_{f \geqslant 0} \left\{ \int A_i^{G_{\ell,s}}(\xi_s, \omega_s) \sqrt{f}(\xi, \omega) d\nu_{\widehat{\theta}}^N(\xi, \omega) + \frac{1}{\gamma N^d} \langle \mathfrak{L}_N \sqrt{f}, \sqrt{f} \rangle \right\} ds \tag{5.4.18}$$

By Lemma 5.C.1, there exist positive constants K_0 and K_1 such that

$$N^2 \langle \mathcal{L}_N \sqrt{f}, \sqrt{f} \rangle + \langle \mathbb{L}_N \sqrt{f}, \sqrt{f} \rangle \leqslant -K_0 N^2 \mathcal{D}_N^0(f) + K_1 N^d.$$

Therefore,

$$\begin{aligned}
&\frac{1}{\gamma N^d} \log \mathbb{E}_{\nu_{\widehat{\theta}}^N}^{N,\widehat{b}} \left(\exp\left(N^d \int_0^T \gamma A_i^{G_{\ell,s}}(\xi_s, \omega_s) ds \right) \right) \\
&\quad \leqslant \int_0^T \sup_f \left\{ \int A_i^{G_{\ell,s}}(\xi_s, \omega_s) f(\xi, \omega) d\nu_{\widehat{\theta}}^N(\xi, \omega) - \frac{1}{\gamma N^{d-2}} \mathcal{D}_N^0(f) \right\} ds + \frac{K_1}{\gamma}
\end{aligned}$$

Now use the change of variables $(\xi', \omega') = (\xi^{x,y}, \omega^{x,y})$ so that

$$\begin{aligned}
&\int A_i^{G_\ell}(\xi, \omega) f(\xi, \omega) d\nu_{\widehat{\theta}}^N(\xi, \omega) \\
&= \frac{1}{N^{d-1}} \int \sum_{j=1}^d \sum_{x \in \Lambda_N} G_\ell(x/N)(\eta_i(x+e_j) - \eta_i(x)) f(\xi, \omega) d\nu_{\widehat{\theta}}^N(\xi, \omega) \\
&= \frac{1}{N^{d-1}} \int \sum_{j=1}^d \sum_{x \in \Lambda_N} G_\ell(x/N) \eta_i(x+e_j) \left(\sum_{u \neq i} \eta_u(x) \right) f(\xi, \omega) d\nu_{\widehat{\theta}}^N(\xi, \omega) \\
&\quad - \frac{1}{N^{d-1}} \int \sum_{j=1}^d \sum_{x \in \Lambda_N} G_\ell(x/N) \eta_i(x) \left(\sum_{u \neq i} \eta_u(x+e_j) \right) f(\xi, \omega) d\nu_{\widehat{\theta}}^N(\xi, \omega) \\
&= \frac{1}{N^{d-1}} \int \sum_{j=1}^d \sum_{x \in \Lambda_N} \sum_{u \neq i} G_\ell(x/N) \eta_u(x+e_j) \eta_i(x) R_{i,u}^{x,y}(\widehat{\theta}) f(\xi^{x,y}, \omega^{x,y}) d\nu_{\widehat{\theta}}^N(\xi, \omega) \\
&+ \frac{1}{N^{d-1}} \int \sum_{j=1}^d \sum_{x \in \Lambda_N} \sum_{u \neq i} G_\ell(x/N) \eta_u(x+e_j) \eta_i(x) \Big(f(\xi^{x,y}, \omega^{x,y}) - f(\xi, \omega) \Big) d\nu_{\widehat{\theta}}^N(\xi, \omega)
\end{aligned}$$

The first term of the right-hand side is of order $O(N^{-1})$ by expanding $R_{i,u}^{x,y}$, while by inequality (5.3.17), the second term is bounded by

$$\begin{aligned}
&N^{2-d} \mathcal{D}_N^0(f) \\
&+ \frac{1}{N^d} \int \sum_{j=1}^d \sum_{x \in \Lambda_N} \sum_{u \neq i} G_\ell^2(x/N) \eta_u(x+e_j) \eta_i(x) \Big(\sqrt{f(\xi^{x,y}, \omega^{x,y})} + \sqrt{f(\xi, \omega)} \Big)^2 d\nu_{\widehat{\theta}}^N(\xi, \omega)
\end{aligned} \tag{5.4.19}$$

To get rid of the second term, note that

$$\int \Big(\sqrt{f(\xi^{x,y},\omega^{x,y})}+\sqrt{f(\xi,\omega)}\Big)^2 d\nu^N_{\hat\theta}(\xi,\omega)$$
$$=\int f(\xi^{x,y},\omega^{x,y})d\nu^N_{\hat\theta}(\xi,\omega)+\int f(\xi,\omega)d\nu^N_{\hat\theta}(\xi,\omega)+\int \sqrt{f(\xi^{x,y},\omega^{x,y})}\sqrt{f(\xi,\omega)}d\nu^N_{\hat\theta}(\xi,\omega)$$

After a change of variable by Lemma 5.A.1, the first and second integrals are equal to a constant times the L^1-norm of f. Use inequality (5.3.17) to divide the third integral into two similar terms. Then, since f is a density with respect to $\nu^N_{\hat\theta}$, for all positive A, (5.4.19) is bounded by

$$N^{2-d}\mathcal{D}^0_N(f)+\frac{C'}{A}\frac{1}{N^d}\sum_{x\in\Lambda_N}G^2_{\ell,t}(x/N)$$

The expression between brackets in (5.4.17) is then bounded by

$$\frac{C'}{AN^d}\sum_{x\in\Lambda_N}G^2_{\ell,t}(x/N)$$

Choose $2C'=A$ to conclude. □

Lemma 5.4.5 (Energy estimate)**.** *Let $Q^{\hat b,*}$ be a limit point of the sequence $(Q^{N,\hat b}_{\mu_N})_{N\geqslant1}$. Then,*

$$Q^{\hat b,*}\Big(L^2([0,T],H^1(\Lambda)\Big)=1 \tag{5.4.20}$$

Proof. Fix $1\leqslant j\leqslant d$. Remark that

$$\lim_{N\to\infty}\int_0^T A_i^{G_t}(\xi_t,\omega_t)dt=\sum_{j=1}^d\int_0^T\int_\Lambda \partial_{e_j}G_{\ell,t}(u)\pi^i_t(u)dt\ du.$$

If $(\mathbb{Q}^{N,\hat b}_{\mu_N})_{N\geqslant1}$ converges weakly to $\mathbb{Q}^{\hat b,*}$, by Lemma 5.4.4,

$$\mathbb{E}^{\hat b,*}\left(\max_{1\leqslant\ell\leqslant k}\Big(\int_0^T\int_\Lambda\partial_{e_j}G_{\ell,s}(u)\pi^i_s(u)du\ ds-\frac12\int_0^T\int_\Lambda G_{\ell,s}(u)^2du\ ds\Big)\right)\leqslant C_0.$$

Since $(G_\ell)_{\ell\geqslant1}$ is dense in $\mathcal{C}^\infty_c([0,T]\times\Lambda;\mathbb{R})$, taking the limit as k goes to infinity, one has by monotone convergence theorem,

$$\mathbb{E}^{\hat b,*}\left(\sup_{G\in\mathcal{C}^\infty_c([0,T]\times\Lambda;\mathbb{R})}\Big(\int_0^T\int_\Lambda\partial_{e_j}G_s(u)\pi^i_s(u)du\ ds-\frac12\int_0^T\int_{\mathbb{T}^d}G_s(u)^2du\ ds\Big)\right)\leqslant C_0.$$

Therefore, for all i, there exists some positive constant C so that for any smooth function $G\in\mathcal{C}^\infty_c([0,T]\times\Lambda,\mathbb{R})$, under $\mathbb{Q}^{\hat b,*}$,

$$\sum_{j=1}^d\int_0^T ds\int_\Lambda du\ \rho_i(s,u)\partial_{e_j}G_s(u)\leqslant\frac12\int_0^T ds\int_\Lambda duG_s(u)^2+C$$

hence, $\hat\rho\in L^2([0,T],H^1(\Lambda))^3$. □

5.4.3 The hydrodynamic limit.

To derive the hydrodynamic behaviour of our system, we divide the proof into several steps :

(1) tightness of the measures $(Q^{N,\widehat{b}}_{\mu_N})_{N\geqslant 1}$;
(2) uniqueness of a weak solution to the hydrodynamic equation (5.2.31) ;
(3) identification of the limit points of $(Q^{N,\widehat{b}}_{\mu_N})_{N\geqslant 1}$ as unique weak solutions of (5.2.31).

The identification of the limit points is provided by the following Proposition and Lemmata.

Lemma 5.4.6 (Tightness). *The sequence $(Q^{N,\widehat{b}}_{\mu_N})_{N\geqslant 1}$ is tight and all its limit points $Q^{\widehat{b},*}$ are concentrated on*

$$Q^{\widehat{b},*}\Big(\widehat{\pi} : 0 \leqslant \widehat{\pi}_t(u) \leqslant 1,\ \widehat{\pi}_t(du) = \widehat{\pi}_t(u)du,\ t \in [0,T]\Big) = 1. \tag{5.4.21}$$

Proof. For this proof, we refer to Chapter 4, indeed to estimate $\langle M^{N,i}\rangle_t$ for the martingale (5.4.1), note that $\widehat{G}$ vanishes at the boundaries on Λ. Therefore, see (5.4.2), the involved generator to derive $\langle M^{N,i}\rangle_t$ is in fact $N^2\mathcal{L}_N + \mathbb{L}_N$. It yields

$$\langle M^{N,i}\rangle_t \leqslant \frac{(C(\lambda_1,\lambda_2,r)\|G_i\|_\infty + C)t}{N^d}$$

which converges to zero as $N \to \infty$. And on the other hand, recall we have (5.4.2) so that $|N^2\mathcal{L}_N\langle \pi^{N,i}_t, G_i\rangle| \leqslant \|\Delta G_i\|_1 + 2\|\nabla G_i\|_1$. □

Denote by ι_ϵ the approximation of the identity

$$\iota_\epsilon = (2\epsilon)^{-d}\mathbf{1}\{[-\epsilon,\epsilon]^d\}.$$

To show $Q^{\widehat{b},*}$ is supported on densities $\widehat{\rho}$ that are weak solutions of (5.2.31).

Lemma 5.4.7 (Identification of limit points). *All limit points $Q^{\widehat{b},*}$ of the sequence $(Q^{N,\widehat{b}}_{\mu_N})_{N\geqslant 1}$ are concentrated on trajectories that are weak solutions of system* (5.2.31).

For further details on the proof, we refer to Chapter 4. The difference here is we need to highlight the replacement of local functions at boundaries. Define the functional $\widehat{B}_\epsilon$ for any function $\widehat{G} \in \mathcal{C}^{1,2}_0([0,T]\times\Lambda;\mathbb{R}^3)$ whose first component is given by

$$\begin{aligned}
B^1_\epsilon(\widehat{\pi}^N_\cdot) \quad := \quad & \langle \pi^{N,1}_T, G_{1,T}\rangle - \langle \pi^{N,1}_0, G_{1,0}\rangle \\
- \quad & \int_0^T \langle \pi^{N,1}_s, \partial_s G_{1,s}\rangle ds - \int_0^T \langle \pi^{N,1}_s, \Delta_N G_{1,s}\rangle ds \\
+ \quad & \int_0^T \sum_{x\in\Gamma^+_N} \partial_{e_1} G_{1,s}(x/N) b_1(x/N) ds - \int_0^T \sum_{x\in\Gamma^-_N} \partial_{e_1} G_{1,s}(x/N) b_1(x/N) ds
\end{aligned}$$

$$
\begin{aligned}
&- \int_0^T \frac{2d\lambda_1}{N^d} \sum_{\Lambda_N} G_{1,s}(x/N)(\pi_s^{N,1} * \iota_\epsilon)(\pi_s^{N,0} * \iota_\epsilon) ds \\
&- \int_0^T \frac{2d\lambda_2}{N^d} \sum_{\Lambda_N} G_{1,s}(x/N)(\pi_s^{N,3} * \iota_\epsilon)(\pi_s^{N,0} * \iota_\epsilon) ds \\
&- r \int_0^T \langle \pi_s^{N,3}, G_{1,s} \rangle ds + \int_0^T (r+1) \langle \pi_s^{N,1}, G_{1,s} \rangle ds
\end{aligned}
$$

while other component are defined the same way. It is enough to treat the case $i = 1$. By Proposition 5.4.1, we may replace local functions of (ξ, ω) in the martingale (5.4.1). Since occupations variables $\eta_i(x)$ are of mean $\eta_i^{\epsilon N}$, resp. $b_i(x/N)$, under the measure $\nu_{\widehat{\eta}^{\epsilon n}}^N$, resp. $\nu_{\widehat{b}}^N$, one has

$$
\overline{\lim_{\epsilon \to 0}} \, \overline{\lim_{N \to \infty}} \, Q_{\mu_N}^{\widehat{b},*} \Big(\sup_{0 \leqslant t \leqslant T} \big| B_\epsilon^1(\widehat{\pi}_t^N \big| > a \Big) = 0.
$$

Notice $\pi_\cdot \mapsto B_\epsilon^1(\widehat{\pi}_T)$ is continuous with respect to the Skorohod topology and let N go to infinity. We conclude using Lemma 5.4.6 and letting ϵ go to zero.

5.5 Empirical currents

In this section, we derive the law of large numbers for the empirical currents stated in Proposition 5.2.1. Recall that for $x \in \Lambda_N$ and $j = 1, \ldots, d$, $W_t^{x,x+e_j}(\eta_i)$ stands for the conservative current of particles of type i across the edge $\{x, x+e_j\}$, and $Q_t^x(\eta_i)$ the total number of particles of type i created minus the total number of particles of type i annihilated at site x before time t. We have the following families of jump martingales (see Lemma 5.B.1 for details) : for all $1 \leqslant j \leqslant d$, $x \in \Lambda_N$,

$$
\begin{aligned}
\widetilde{W}_t^{x,x+e_j}(\eta_i) &= W_t^{x,x+e_j}(\eta_i) - N^2 \int_0^t \Big(\eta_{i,s}(x)(1-\eta_{i,s}(x+e_j)) \\
&\qquad - (1-\eta_{i,s}(x))\eta_{i,s}(x+e_j) \Big) ds \qquad (5.5.1)
\end{aligned}
$$

with quadratic variation (because $J_t^{x,x+e_j}(\eta_i)$ and $J_t^{x+e_j,x}(\eta_i)$ have no common jump)

$$
\begin{aligned}
\langle \widetilde{W}^{x,x+e_j}(\eta_i) \rangle_t &= \langle \widetilde{J}^{x,x+e_j}(\eta_i) \rangle_t + \langle \widetilde{J}^{x+e_j,x}(\eta_i) \rangle_t \qquad (5.5.2) \\
&= N^2 \int_0^t \Big(\eta_{i,s}(x)(1-\eta_{i,s}(x+e_j)) + (1-\eta_{i,s}(x))\eta_{i,s}(x+e_j) \Big) ds
\end{aligned}
$$

and

$$
\widetilde{Q}_t^x(\eta_i) = Q_t^x(\eta_i) - \int_0^t \tau_x f_i(\xi_s, \omega_s) ds \qquad (5.5.3)
$$

where $\widehat{f} = (f_1, f_2, f_3) : \widehat{\Sigma}_N \to \mathbb{R}^3$ is defined by

$$\begin{cases} f_1(\xi,\omega) &= \beta_N(0,\xi,\omega)\eta_0(0) + \eta_3(0) - (r+1)\eta_1(0)\,, \\ f_2(\xi,\omega) &= r\eta_0(0) + \eta_3(0) - \beta_N(0,\xi,\omega)\eta_2(0) - \eta_2(0)\,, \\ f_3(\xi,\omega) &= \beta_N(0,\xi,\omega)\eta_2(0) + r\eta_1(0) - 2\eta_3(0)\,. \end{cases} \tag{5.5.4}$$

with quadratic variations

$$\begin{cases} \langle \widetilde{Q}^x(\eta_1)\rangle_t &= \int_0^t \tau_x\Big(\beta_N(0,\xi_s,\omega_s)\eta_{0,s}(0) + \eta_{3,s}(0) + (r+1)\eta_{1,s}(0)\Big)ds \\ \langle \widetilde{Q}^x(\eta_2)\rangle_t &= \int_0^t \tau_x\Big(r\eta_{0,s}(0) + \eta_{3,s}(0) + \beta_N(0,\xi_s,\omega_s)\eta_{2,s}(0) + \eta_{2,s}(0)\Big)ds \\ \langle \widetilde{Q}^x(\eta_3)\rangle_t &= \int_0^t \tau_x\Big(\beta_N(0,\xi_s,\omega_s)\eta_{2,s}(0) + r\eta_{1,s}(0) + 2\eta_{3,s}(0)\Big)ds \end{cases} \tag{5.5.5}$$

Proof of Proposition 5.2.1. Given a smooth continuous vector field $\mathbf{G} = (G_1, ..., G_d) \in \mathcal{C}_c^\infty(\Lambda, \mathbb{R}^d)$, after definition (5.2.35), sum the martingale (5.5.1) over $\{x, x+e_j \in \Lambda_N\}$ to get the martingale $\widetilde{\mathbf{M}}_t^G$, given by

$$\begin{aligned} \widetilde{\mathbf{M}}_t^G(\eta_i) &= \sum_{j=1}^d \Bigg(\langle W_{j,t}^N(\eta_i), G_j\rangle \\ &\quad - \frac{N^2}{N^{d+1}} \sum_{x,x+e_j\in\Lambda_N} \int_0^t G_j(x/N)\Big(\eta_{i,s}(x) - \eta_{i,s}(x+e_j)\Big)ds\Bigg) \\ &= \langle \mathbb{W}_t^N(\eta_i), \mathbf{G}\rangle - \frac{1}{N^d}\sum_{j=1}^d \sum_{x\in\Lambda_N} \int_0^t \partial_{x_j} G_j(x/N)\eta_{i,s}(x)ds + O(N^{-1}) \\ &= \langle \mathbb{W}_t^N(\eta_i), \mathbf{G}\rangle - \sum_{j=1}^d \langle \pi_s^{N,i}, \partial_{x_j} G_j\rangle + O(N^{-1}) \end{aligned}$$

where we did a Taylor expansion. Relying on (5.5.2), the expectation of $\langle \widetilde{\mathbf{M}}^G\rangle_t$ vanishes when $N \to \infty$, so that by Doob's martingale inequality,

$$\lim_{N\to\infty} \mathbb{P}_{\mu_N}^{N,\hat{b}}\Big[\sup_{0\leqslant t\leqslant T} |\widehat{\widetilde{\mathbf{M}}_t^G}| > \delta\Big] \;=\; 0\,,$$

for any $\delta > 0$. Using that the empirical density $\widehat{\pi}$ converges towards the solution of (5.2.31), this concludes the law of large numbers (5.2.36) for the current $\mathbb{W}_T^N$.

Fix a smooth vector field $\widehat{H} = (H_1, H_2, H_3) \in \mathcal{C}_c^\infty(\Lambda, \mathbb{R}^3)$. Sum (5.5.3) over $x \in \Lambda_N$ to get the martingale

$$\widetilde{\mathbf{N}}_t^H(\eta_i) = \langle Q_t^N(\eta_i), H_i\rangle - \frac{1}{N^d}\sum_{x\in\Lambda_N}\int_0^t H_i(x/N)\tau_x f_i(\xi_s,\omega_s)ds$$

Relying on (5.5.5), the expectation of its quadratic variation vanishes as $N \to \infty$ as well. Use the Replacement lemma to express $\widetilde{\mathbf{N}}_t^H(\eta_i)$ with functionals of the density fields and conclude to (5.2.37) by Doob's martingale inequality having for any $\delta > 0$,

$$\lim_{N\to\infty} \mathbb{P}_{\mu_N}^{N,\hat{b}}\Big[\sup_{0\leqslant t\leqslant T} |\widehat{\widetilde{\mathbf{N}}_t^H}| > \delta\Big] = 0.$$

□

5.6 Hydrodynamics in infinite volume

In this section, we derive the hydrodynamic limit in infinite volume of Theorem 5.2.2.

5.6.1 Replacement lemma

To close the equations in the expression of martingales, we state here the replacement lemma for the infinite volume. It relies on uniform upper bounds on the entropy production and the Dirichlet form given by Theorem 5.2.1 and proved in Section 5.3.

We shall make use of Theorem 5.2.1, with a slight difference : we consider here for any $n \geqslant 1$, a large finite box $B_n = \{-n, ..., n\}^d$ (instead of $\Lambda_{N,n} = \{-N, ..., N\} \times \{-n, ..., n\}^{d-1}$), since we do not require boundary conditions. Indeed, to reach $\mathbb{Z}^d$, in the proof of Lemma 5.2.1 we need to expand the box B_n over B_{n+1} in each direction $(e_1, ..., e_d)$ so that in our estimates : n^{d-1} is replaced by n^d. Therefore, the result of Theorem 5.2.1 still holds.

Lemma 5.6.1 (replacement lemma). *For any $G \in \mathcal{C}_c^\infty([0,T] \times \overline{\Lambda}, \mathbb{R})$,*

$$\overline{\lim_{\epsilon\to 0}}\ \overline{\lim_{N\to\infty}}\ \mathbb{E}_{\mu_N}^N\Big(\frac{1}{N^d}\sum_{x\in\mathbb{Z}^d}\int_0^T |G_s(x/N)|\tau_x V_{\epsilon N}(\xi_s,\omega_s)ds\Big) = 0, \tag{5.6.1}$$

where $V_{\epsilon N}(\xi,\omega)$ was defined in (5.4.6)

Proof. Let $M > 0$ so that G has compact support contained in $[-M, M]^d$. Therefore,

$$\begin{aligned}&\mathbb{E}_{\mu_N}^N\Big(\frac{1}{N^d}\sum_{x\in\mathbb{Z}^d}\int_0^T |G_s(x/N)|\tau_x V_{\epsilon N}(\xi_s,\omega_s)ds\Big)\\ &\qquad\leqslant \|G\|_\infty \mathbb{E}_{\mu_N}^N\left(\frac{1}{N^d}\sum_{x\in B_{MN}}\int_0^T \tau_x V_{\epsilon N}(\xi_s,\omega_s)ds\right).\end{aligned} \tag{5.6.2}$$

Now, $V_{\epsilon N}(\xi,\omega)$ depends on configurations only through occupations variables $\{(\xi,\omega)(x) : x \in B_{MN}\}$, by Fubini's theorem and Theorem 5.2.1, there exists some positive constant C_1 such that the expectation in (5.6.2) is bounded by

$$\frac{T}{N^d}\int \sum_{x\in B_{MN}} \tau_x V_{\epsilon N}(\xi,\omega)\bar{f}^T(\xi,\omega)d\nu^N_{\hat{\theta}_a}(\xi,\omega) - \gamma T N^{2-d}\mathcal{D}^0_{(M+2)N}(\bar{f}^T) + \gamma C_1,$$

for all positive γ, where $\bar{f}^T = T^{-1}\int_0^T f^s_{(M+2)N}ds$, with $f^t_{(M+2)N}$ standing for the density of $\mu_N(t)$ with respect to $\nu^{(M+2)N}_{\hat{\theta}}$, the restriction of $\nu_{\hat{\theta}}$ to the box B_{NM}. It thus remains to show that

$$\overline{\lim_{\epsilon\to 0}}\ \overline{\lim_{N\to\infty}}\ \sup_f \left\{ \int \frac{1}{N^d}\sum_{x\in B_{MN}} \tau_x V_{\epsilon N}(\xi,\omega)\overline{f}^T(\xi,\omega)d\nu^{(M+2)N}_{\hat{\theta}}(\xi,\omega) - \gamma N^{2-d}\mathcal{D}^0_{(M+2)N}(\overline{f}^T)\right\} = 0.$$

This limit is a consequence of the one and two blocks estimates (5.4.2)–(5.4.3), for which we refer to Chapter 4 since we reduced ourselves to a finite volume and conclude by letting γ go to 0. $\square$

5.6.2 The hydrodynamic limit

To conclude to the hydrodynamic behaviour of our system, we still need to prove : tightness of the measures $(Q^{N,\hat{b}}_{\mu_N})_{N\geqslant 1}$; identification of the limit points of $(Q^{N,\hat{b}}_{\mu_N})_{N\geqslant 1}$; uniqueness of weak solutions of the hydrodynamic equation.

The two first steps are analogous to the proofs done in finite volume, we refer the reader to Chapter 4 for details. Though, we prove the uniqueness of weak solutions for the generalized contact process in infinite volume with stochastic reservoirs in Section 5.7, the method yields to prove the uniqueness of weak solutions of the system (5.2.38), this is given by Proposition 5.2.2 whose proof is postponed to Section 5.7.

5.7 Uniqueness of weak solutions

To conclude, we derive in this section the uniqueness of the weak solutions of Section 5.2.

5.7.1 Uniqueness in finite volume

Proof of Lemma 5.2.2. Let $\hat{\rho}^{(1)}$ and $\hat{\rho}^{(2)}$ be two weak solutions of 5.2.31 satisfying (B1) and (B2), starting from the initial profile $\hat{\gamma}$. For a given $\delta > 0$, denote by A_δ the regularized absolute value function

$$A_\delta(u) := \frac{u^2}{2\delta}\mathbf{1}\{|u| \leqslant \delta\} + \Big(|u| - \frac{\delta}{2}\Big)\mathbf{1}\{|u| > \delta\}.$$

Since $\mathcal{C}_c^\infty(\Lambda;\mathbb{R})$ is dense in $H^1(\Lambda)$, by approximating A_δ by smooth functions and using (B2), we get (cf. [28])

$$\begin{aligned}
&\sum_i \partial_t \int A_\delta\Big(\rho_i^{(1)}(t,u) - \rho_i^{(2)}(t,u)\Big)du \\
&\quad= \sum_i \int A'_\delta\Big(\rho_i^{(1)}(t,u) - \rho_i^{(2)}(t,u)\Big)\partial_t\Big(\rho_i^{(1)}(t,u) - \rho_i^{(2)}(t,u)\Big)du \\
&\quad= -\sum_i \int \Big\{\nabla A'_\delta\Big(\rho_i^{(1)}(t,u) - \rho_i^{(2)}(t,u)\Big)\Big(\nabla\rho_i^{(1)}(t,u) - \nabla\rho_i^{(2)}(t,u)\Big\}du \\
&\qquad+ \sum_i \int A'_\delta\Big(\rho_i^{(1)}(t,u) - \rho_i^{(2)}(t,u)\Big)\Big(F_i(\widehat{\rho}^{(1)}(t,u)) - F_i(\rho^{(2)}(t,u))\Big)du \\
&\quad= -\frac{1}{\delta}\sum_i \int \nabla\Big(\rho_i^{(1)}(t,u) - \rho_i^{(2)}(t,u)\Big).\nabla\Big(\rho_i^{(1)}(t,u) - \rho_i^{(2)}(t,u)\Big)\mathbf{1}_{V_\delta}du \\
&\qquad+ \sum_i \int \Big(F_i(\widehat{\rho}^{(1)}(t,u)) - F_i(\rho^{(2)}(t,u))\Big)\Big(\frac{\rho_i^{(1)}(t,u) - \rho_i^{(2)}(t,u)}{\delta}\mathbf{1}_{V_\delta} + \mathbf{1}_{V_\delta^c}\Big)du \\
&\quad\leqslant -\frac{1}{\delta}\sum_i \int \big\|\nabla\big(\rho_i^{(1)}(t,u) - \rho_i^{(2)}(t,u)\big)\big\|^2\mathbf{1}_{V_\delta}du \\
&\qquad+ \sum_i \int \Big|F_i(\widehat{\rho}^{(1)}(t,u)) - F_i(\rho^{(2)}(t,u))\Big|du
\end{aligned}$$

where $V_\delta = \{(t,x) \in [0,T]\times\Lambda : |\widehat{\rho}^{(1)} - \widehat{\rho}^{(2)}| \leqslant \delta\}$. Remark now that $\widehat{F}$ is Lipschitz,

$$|F_i(\widehat{\rho}^{(1)}) - F_i(\widehat{\rho}^{(2)})| \leqslant C(\lambda_1,\lambda_2,r)\sum_i |\rho_i^{(1)} - \rho_i^{(2)}|, \text{ for all } i = 1,2,3.$$

Therefore,

$$\sum_i \partial_t \int A_\delta\Big(\rho_i^{(1)}(t,u) - \rho_i^{(2)}(t,u)\Big)du \leqslant C(\lambda_1,\lambda_2,r)\sum_i \int \Big|\rho_i^{(1)}(t,u) - \rho_i^{(2)}(t,u)\Big|du.$$

One concludes by letting $\delta \downarrow 0$ and using Gronwall's inequality. □

5.7.2 Uniqueness in infinite volume with boundaries

Proof of Proposition 5.2.2. The proof follows the arguments in [61] adapted to the our case. For $u = (u_1, \cdots, u_d) \in \Lambda^\infty$, denote by $\check{u} = (u_2, \cdots, u_d) \in \mathbb{R}^{d-1}$, so that $u = (u_1, \check{u})$.

Denote by $L^2((-1,1))$ the Hilbert space on the one-dimentional bounded interval $(-1,1)$ equipped with the inner product,

$$\langle \varphi, \psi \rangle_2 = \int_{-1}^{1} \varphi(u_1)\,\overline{\psi(u_1)}\,du_1\ ,$$

where, for $z \in \mathbb{C}$, $\bar{z}$ is the complex conjugate of z and $|z|^2 = z\bar{z}$. The norm of $L^2((-1,1))$ is denoted by $\|\cdot\|_2$.

Let $H^1((-1,1))$ be the Sobolev space of functions φ with generalized derivatives $\partial_{u_1}\varphi$ in $L^2((-1,1))$. $H^1((-1,1))$ endowed with the scalar product $\langle \cdot, \cdot \rangle_{1,2}$, defined by

$$\langle \varphi, \psi \rangle_{1,2} = \langle \varphi, \psi \rangle_2 + \langle \partial_{u_1}\varphi\,,\,\partial_{u_1}\psi \rangle_2\ ,$$

is a Hilbert space. The corresponding norm is denoted by $\|\cdot\|_{1,2}$.

Consider the following classical boundary-eigenvalue problem for the Laplacian :

$$\begin{cases} -\Delta\varphi = \alpha\varphi\,, \\ \varphi \in H_0^1((-1,1))\,. \end{cases} \tag{5.7.1}$$

From the Sturm–Liouville theorem (cf. [77]), one can construct for the problem (5.7.1) a countable system $\{\varphi_n, \alpha_n : n \geqslant 1\}$ of eigensolutions which contains all possible eigenvalues. The set $\{\varphi_n : n \geqslant 1\}$ of eigenfunctions forms a complete orthonormal system in the Hilbert space $L^2((-1,1))$. Moreover each φ_n belong to $H_0^1((-1,1))$ and the set $\{\varphi_n/\alpha_n^{1/2} : n \geqslant 1\}$ is a complete orthonormal system in the Hilbert space $H_0^1((-1,1))$. Hence, a function ψ belongs to $L^2((-1,1))$ if and only if

$$\psi \;=\; \lim_{n\to\infty}\sum_{k=1}^{n}\langle \psi, \varphi_k \rangle_2\,\varphi_k$$

in $L^2((-1,1))$. In this case, for each $\psi_1, \psi_2 \in L^2((-1,1))$

$$\langle \psi_1, \psi_2 \rangle_2 \;=\; \sum_{k=1}^{\infty}\langle \psi_1, \varphi_k \rangle_2\,\overline{\langle \psi_2, \varphi_k \rangle_2}\,.$$

Furthermore, a function ψ belongs to $H_0^1((-1,1))$ if and only if

$$\psi \;=\; \lim_{n\to\infty}\sum_{k=1}^{n}\langle \psi, \varphi_k \rangle_2\,\varphi_k$$

in $H_0^1((-1,1))$, and

$$\langle \psi_1, \psi_2 \rangle_{1,2} \;=\; \sum_{k=1}^{\infty}\alpha_k\langle \psi_1, \varphi_k \rangle_2\,\overline{\langle \psi_2, \varphi_k \rangle_2} \tag{5.7.2}$$

for all ψ_1, ψ_2 in $H_0^1((-1,1))$. One can easily check that in our case, $\alpha_n = n^2\pi^2$ and $\varphi_n(u_1) = \sin(n\pi u_1)$, $n \in \mathbb{N}$.

Fix $T > 0$, define the heat Kernel on the the time interval $(0, T]$ defined by the following expression

$$p_1(t, u_1, v_1) \;=\; \sum_{n \geqslant 1} e^{-\alpha_n t} \varphi_n(u_1) \overline{\varphi_n(v_1)}\,,\; t \in [0, T]\,,\; u_1, v_1 \in [-1, 1]\,.$$

Let $g \in \mathcal{C}_c^0((-1,1);\mathbb{R})$ and denote by $\delta.$ the Dirac function. The heat Kernel p_1 is such that $p_1(0, u_1, v_1) = \delta_{u_1 - v_1}$, $p \in \mathcal{C}^\infty((0,T] \times (-1,1) \times (-1,1);\mathbb{R})$ and the function defined via the convolution operator :

$$\varphi_1(t, u_1) := (p_1 \star g)(t, u_1) = \int_{-1}^{1} p_1(t, u_1, v_1) g(v_1) dv_1$$

solves the following boundary value problem

$$\left\{ \begin{array}{l} \partial_t \varphi = \partial^2_{u_1} \varphi\,, \\ \varphi(0, \cdot) = \; g(\cdot)\,, \\ \varphi(t, \cdot) \in H^1_0((-1,1)) \;\text{ for } 0 < t \leqslant T\;. \end{array} \right. \tag{5.7.3}$$

Let $\check{p}$ be the heat kernel for $(t, \check{u}, \check{v}) \in (0, T) \times \mathbb{R}^{d-1} \times \mathbb{R}^{d-1}$

$$\check{p}(t, \check{u}, \check{v}) \;=\; \big(4\pi t\big)^{-(d-1)/2} \exp\left\{ -\frac{1}{4t} \sum_{k=2}^{d} (u_k - v_k)^2 \right\}.$$

For each function $\check{f} \in \mathcal{C}_c(\mathbb{R}^{d-1};\mathbb{R})$

$$\check{h}^{\check{f}}_t(t, \check{u}) := (\check{p} \star \check{f})(t, \check{u}) = \int_{\mathbb{R}^{d-1}} \check{p}(t, \check{u}, \check{v}) \check{f}(\check{v}) d\check{v}\,.$$

It is known that $\check{h}^f_t$ solves the equation $\partial_t \check{\rho} = \Delta \check{\rho}$, $\check{\rho}_0 = f$, on $(0, t] \times \mathbb{R}^{d-1}$. Moreover $\check{h} \in \mathcal{C}^\infty((0,T] \times \mathbb{R}^{d-1};\mathbb{R})$.

For a positive time $t \in (0, T]$, $\widehat{f} = (f_1, f_2, f_3) \in \mathcal{C}_c(\overline{\Lambda^\infty};\mathbb{R}^3)$ and $\varepsilon > 0$ small enough, let $\mathcal{H}^f_{t,\varepsilon} : [0, t] \times \Lambda^\infty \longrightarrow \mathbb{R}$ be defined by

$$\mathcal{H}^{\widehat{f}}_{t,\varepsilon}(s, u) := \sum_{i=1}^{3} \mathcal{H}^{f_i}_{t,\varepsilon}(s, u) := \sum_{i=1}^{3} \big(p * f_i\big)(t + \epsilon - s, u),$$

where p is the heat kernel on $(0, T] \times \Lambda^\infty \times \Lambda^\infty$ given by

$$p(t, u, v) = p_1(t, u_1, v_1) \check{p}(t, \check{u}, \check{v}).$$

It is easy to check that $\mathcal{H}^f_{t,\varepsilon}$ solves the equation $\partial_t \rho = \Delta \rho$ on $(0, t] \times \mathbb{R}^d$, $\rho_0 = f$.

Consider $\widehat{\rho}^{(1)} = (\rho_1^{(1)}, \rho_2^{(1)}, \rho_3^{(1)})$ and $\widehat{\rho}^{(2)} = (\rho_1^{(2)}, \rho_2^{(2)}, \rho_3^{(2)})$ two weak solutions of (5.2.31) associated to an initial profile $\widehat{\gamma} = (\gamma_1, \gamma_2, \gamma_3) : \Lambda^\infty \to [0, 1]^3$. Set $\overline{m}_i = \rho_i^{(1)} - \rho_i^{(2)}$,

$1 \leqslant i \leqslant 3$. We shall prove below that for any function $m(\cdot,\cdot) \in L^\infty([0,T] \times \Lambda^\infty)$ and each $i \leqslant i \leqslant d$,

$$\int_0^t ds \left| \int_{\Lambda^\infty} m(s,u) \mathcal{H}^{f_i}_{t,\varepsilon}(s,v) dv \right| ds \;\leqslant\; C_1 t \, \|m\|_\infty \|f_i\|_1, \tag{5.7.4}$$

for some positive constant C_1, where for a trajectory $m : [0,t] \times \Lambda^\infty \to \mathbb{R}$, $\|m\|_\infty = \|m\|_{L^\infty([0,t]\times\Lambda^\infty)}$ stands for the infinite norm in $L^\infty([0,t] \times \Lambda^\infty)$.

On the other hand, from the fact that $\rho_i^{(1)}, \rho_i^{(2)}$, $1 \leqslant i \leqslant 3$ are in $L^\infty([0,T] \times \Lambda^\infty)$, it follows that there exists a positive constant C_2 such that, for almost every $(s,u) \in [0,t] \times \Lambda^\infty$, for every $1 \leqslant i \leqslant 3$,

$$\left| F_i(\rho_i^{(1)}(s,u)) - F_i(\rho_i^{(2)}(s,u)) \right| \leqslant C_2 \sum_{i=1}^3 \|\rho_i^{(1)} - \rho_i^{(2)}\|_\infty .$$

Since $\widehat{\rho}^{(1)}$ and $\widehat{\rho}^{(2)}$ are two weak solutions of (5.2.31), we obtain by (5.7.4) that for all $0 \leqslant \tau \leqslant t$, $1 \leqslant i,k \leqslant 3$

$$\begin{aligned} \left| \left\langle \overline{m}_i(\tau,.), \mathcal{H}^{f_k}_{\tau,\varepsilon}(\tau,.) \right\rangle \right| &= \sum_{i=1}^3 \left| \int_0^\tau \left\langle F_i(\widehat{\rho}^{(1)}) - F_i(\widehat{\rho}^{(2)}), \mathcal{H}^{f_k}_{\tau,\varepsilon}(\tau,.) \right\rangle \right| \\ &\leqslant C_1' t \left(\sum_{i=1}^3 \|\rho_i^{(1)} - \rho_i^{(2)}\|_\infty \right) \|f_k\|_1 , \end{aligned}$$

for $C_1' = C_1 C_2$.

By observing that $p(\varepsilon,\cdot,\cdot)$ is an approximation of the identity in ε, we obtain by letting $\varepsilon \downarrow 0$,

$$\left| \left\langle \overline{m}_i(\tau,.), f_k \right\rangle \right| \;\leqslant\; C_1' t \left(\sum_{i=1}^3 \|\rho_i^{(1)} - \rho_i^{(2)}\|_\infty \right) \|f_k\|_1 . \tag{5.7.5}$$

We claim that $\overline{m}_i \in L^\infty([0,t] \times \Lambda^\infty)$ and

$$\|\overline{m}_i\|_\infty \;\leqslant\; C_1' \, t \left(\sum_{i=1}^3 \|\rho_i^{(1)} - \rho_i^{(2)}\|_\infty \right) . \tag{5.7.6}$$

Indeed (cf. [67], [61]), denote by $R(t) = \sum_{i=1}^3 \|\rho_i^{(1)} - \rho_i^{(2)}\|_\infty$, by (5.7.5), for any open set U of Λ^∞ with finite Lebesgue measure $\lambda(U)$, we have for all $0 \leqslant \tau \leqslant t$,

$$\int_U \overline{m}_i(\tau,u) du \leqslant C_1' \, t \, R(t) \lambda(U). \tag{5.7.7}$$

Fix $0 < \delta < 1$. For any open set U of Λ^∞ with finite Lebesgue measure and for $0 \leqslant \tau \leqslant t$ let

$$B^U_{\delta,\tau} = \left\{ u \in U \;:\; \overline{m}_i(\tau,u) > C_1' \, t \, R(t)(1+\delta) \right\}.$$

Suppose that $\lambda(B^U_{\delta,\tau}) > 0$, there exists an open set V, such that, $B^U_{\delta,\tau} \subset V$ and $\lambda\big(V \backslash B^U_{\delta,\tau}\big) \leqslant \lambda(V)\frac{\delta}{2}$ and we have

$$\begin{aligned}
\lambda(V)\big(C_1'\, t\, R(t)\big) &< \lambda(V)\big(C_1'\, t\, R(t)\big)(1+\delta)(1-\delta/2) \\
&= \big(C_1'\, t\, R(t)\big)(1+\delta)\big(\lambda(V) - \lambda(V)\delta/2\big) \\
&\leqslant \big(C_1' t\, R(t)\big)(1+\delta)\big(\lambda(V) - \lambda\big(V \backslash B^U_{\delta,\tau}\big)\big) \\
&= \big(C_1'\sqrt{t} R(t)\big)(1+\delta)\lambda\big(B^U_{\delta,\tau}\big) \\
&< \int_{B^U_{\delta,\tau}} \overline{m}_i(\tau, x)dx\,.
\end{aligned}$$

Thus, from (5.7.7) and since $B^U_{\delta,\tau} \subset V$, we get

$$\begin{aligned}
\lambda(V)\big(C_1'\, t\, R(t)\big) &< \int_V \overline{m}_i(\tau, x)dx \\
&\leqslant \big(C_1'\, t\, R(t)\big)\lambda(V)\,,
\end{aligned}$$

which leads to a contradiction.

By the arbitrariness of $0 < \delta < 1$ we obtain that if U is any open set of Λ^∞ with $\lambda(U) < \infty$,

$$\lambda\Big(\Big\{u \in U \;:\; \overline{m}_i(\tau, u) > C_1'\, t\, R(t)\Big\}\Big) = 0.$$

This implies

$$\overline{m}_i(\tau, x) \leqslant C_1'\, t\, R(t) \qquad \text{a.e. in } \Lambda^\infty$$

and concludes the proof of (5.7.6) by the arbitrariness of $\tau \in [0, t]$.

We now turn to the proof of the uniqueness, from (5.7.6),

$$\|\overline{m}_i\|_\infty \;\leqslant\; C_1'\, t\, \Big(\sum_{j=1}^{3} \|\overline{m_j}\|_\infty\Big)\,,$$

and then

$$R(t) \leqslant 3C_1'\, t\, R(t)\,.$$

Choosing $t = t_0$ such that $3C_1'\, t_0 \; < 1$, this gives uniqueness in $[0, t_0] \times \Lambda^\infty$. To conclude the proof we have just to repeat the same arguments in $[t_0, 2t_0]$, and in each interval $[kt_0, (k+1)t_0]$, $k \in \mathbb{N}$, $k > 1$.

it remains to prove inequality (5.7.4). From Fubini's Theorem, we have

$$
\begin{aligned}
&\int_0^t \left| \int_{\Lambda^\infty} m(s,u)\mathcal{H}^{f_i}_{t,\varepsilon}(s,u)du \right| ds \\
&\leqslant \int_0^t ds \int_{\mathbb{R}^{d-1}} d\check{v} \int_{\mathbb{R}^{d-1}} d\check{u} \Bigg| \sum_{n\geqslant 1} e^{-n^2\pi^2(t+\varepsilon-s)} \int_{-1}^1 dv_1 \Big\{ \sin(n\pi v_1) f_i(v_1,\check{v}) \Big\} \\
&\qquad\qquad \times \int_{-1}^1 du_1 \Big\{ \sin(n\pi u_1)\check{p}(t+\varepsilon-s,\check{u},\check{v}) m(s,u_1,\check{u}) \Big\} \Bigg| \\
&\leqslant \int_0^t ds \int_{\mathbb{R}^{d-1}} d\check{v} \int_{\mathbb{R}^{d-1}} d\check{u}\, \check{p}(t+\varepsilon-s,\check{u},\check{v}) \\
&\qquad\qquad \times \left| \sum_{n\geqslant 1} \Big\langle \varphi_n, m(s,(\cdot,\check{u})) \Big\rangle \times \Big\langle \varphi_n, f_i(\cdot,\check{v}) \Big\rangle \right| \\
&\leqslant \int_0^t ds \int_{\Lambda^\infty} du \int_{\Lambda^\infty} dv \Big\{ |m(s,u)|\, |f_i(v)|\, \check{p}(t+\varepsilon-s,\check{u},\check{v}) \Big\} \\
&\leqslant 4\,t\, \|m\|_\infty \|f_i\|_1\,,
\end{aligned}
$$

where we used the fact that $\check{p}(s,\cdot,\cdot)$ is a probability kernel in $\mathbb{R}^{d-1}$ for all $s>0$. □

5.A Changes of variables formulas

In the following, one states useful formula concerning change of variables with respect to a varying smooth profile. It is convenient to use the form (5.2.17) of the reference measure.

Lemma 5.A.1. *For* $(i,j) \in \{0,1,2,3\}^2, i \neq j$,

$$
\begin{aligned}
&\int \eta_i(x)\eta_j(y) f(\xi^{x,y},\omega^{x,y}) d\nu^N_{\hat\theta}(\xi,\omega) \\
&\qquad = \int \eta_j(x)\eta_i(y) e^{(\vartheta_j(y/N)-\vartheta_j(x/N))-(\vartheta_i(y/N)-\vartheta_i(x/N))} f(\xi,\omega) d\nu^N_{\hat\theta}(\xi,\omega)
\end{aligned}
$$

Proof. Let us detail the change of variable when $(i,j)=(1,2)$, the other ones are similar. Posing $(\xi',\omega') = (\xi^{x,y},\omega^{x,y})$ one has,

$$
\begin{aligned}
&\int \eta_1(x)\eta_2(y) f(\xi^{x,y},\omega^{x,y}) d\nu^N_{\hat\theta}(\xi,\omega) \\
&= \int \xi(x)(1-\omega(x))(1-\xi(y))\omega(y) f(\xi^{x,y},\omega^{x,y}) d\nu^N_{\hat\theta}(\xi,\omega) \\
&= \int (1-\xi'(x))\omega'(x)\xi'(y)(1-\omega'(y)) f(\xi',\omega') \frac{d\nu_{\hat\theta}(\xi'^{x,y},\omega'^{x,y})}{d\nu_{\hat\theta}(\xi',\omega')} d\nu^N_{\hat\theta}(\xi',\omega')
\end{aligned}
$$

$$= \int \eta_2'(x)\eta_1'(y) f(\xi',\omega') \frac{d\nu_{\hat\theta}(\xi'^{x,y},\omega'^{x,y})}{d\nu_{\hat\theta}(\xi',\omega')} d\nu_{\hat\theta}^N(\xi',\omega')$$

but

$$\frac{\nu_{\hat\theta}^N(\xi^{x,y},\omega^{x,y})}{\nu_{\hat\theta}^N(\xi,\omega)} = \exp\left\{ \sum_{\ell=0}^{3} \Big(\vartheta_\ell(x/N) - \vartheta_\ell(y/N)\Big)\Big(\eta_\ell(y) - \eta_\ell(x)\Big) \right\}$$

so that

$$\int \eta_1(x)\eta_2(y) f(\xi^{x,y},\omega^{x,y}) d\nu_{\hat\theta}^N(\xi,\omega)$$
$$= \int \eta_2(x)\eta_1(y) e^{(\vartheta_2(y/N)-\vartheta_2(x/N))-(var\theta_1(y/N)-\vartheta_1(x/N))} f(\xi,\omega) d\nu_{\hat\theta}^N(\xi,\omega)$$

□

Lemma 5.A.2. *(i) for each* $(i,j) \in \{(1,2),(2,1),(3,0),(0,3)\}$,

$$\int \eta_i(x) f(\sigma^x\xi,\sigma^x\omega) d\nu_{\hat\theta}^N(\xi,\omega) = \int \eta_j(x) e^{(\vartheta_i(x/N)-\vartheta_j(x/N))} f(\xi,\omega) d\nu_{\hat\theta}^N(\xi,\omega)$$

(ii) for each $(i,j) \in \{(1,0),(0,1),(3,2),(2,3)\}$,

$$\int \eta_i(x) f(\sigma^x\xi,\omega) d\nu_{\hat\theta}^N(\xi,\omega) = \int \eta_j(x) e^{(\vartheta_i(x/N)-\vartheta_j(x/N))} f(\xi,\omega) d\nu_{\hat\theta}^N(\xi,\omega)$$

(iii) for each $(i,j) \in \{(1,3),(3,1),(2,0),(0,2)\}$,

$$\int \eta_i(x) f(\xi,\sigma^x\omega) d\nu_{\hat\theta}^N(\xi,\omega) = \int \eta_j(x) e^{(\vartheta_i(x/N)-\vartheta_j(x/N))} f(\xi,\omega) d\nu_{\hat\theta}^N(\xi,\omega)$$

Proof. Let us show the lemma for (i) when $(i,j) = (1,2)$. By the change of variables $(\xi',\omega') = (\sigma^x\xi,\sigma^x\omega)$ we have

$$\int \eta_1(x) f(\sigma^x\xi,\sigma^x\omega) d\nu_{\hat\theta}^N(\xi,\omega)$$
$$= \int \xi(x)(1-\omega(x)) f(\sigma^x\xi,\sigma^x\omega) d\nu_{\hat\theta}^N(\xi,\omega)$$
$$= \int (1-\xi'(x))\omega'(x) f(\xi',\omega') \frac{d\nu_{\hat\theta}(\sigma^x\xi',\sigma^x\omega')}{d\nu_{\hat\theta}(\xi',\omega')} d\nu_{\hat\theta}^N(\xi',\omega')$$

but

$$\frac{\nu_{\hat\theta}^N(\sigma^x\xi,\sigma^x\omega)}{\nu_{\hat\theta}^N(\xi,\omega)} = \exp\left\{ \Big(\vartheta_1(x/N) - \vartheta_2(x/N)\Big)\Big(\eta_2(x) - \eta_1(x)\Big) \right.$$

$$+\Big(\vartheta_3(x/N)-\vartheta_0(x/N)\Big)\Big(\eta_0(x)-\eta_3(x)\Big)\Bigg\},$$

so that

$$\int \eta_1(x) f(\sigma^x\xi,\sigma^x\omega)d\nu_{\hat\theta}(\xi,\omega)=\int \eta_2(x)e^{(\vartheta_1(x/N)-\vartheta_2(x/N))}f(\xi,\omega)d\nu^N_{\hat\theta}(\xi,\omega)$$

Deduce *(ii)* and *(iii)* similarly by computing respectively

$$\frac{\nu^N_{\hat\theta}(\sigma^x\xi,\omega)}{\nu^N_{\hat\theta}(\xi,\omega)}=\exp\Bigg\{\Big(\vartheta_1(x/N)-\vartheta_0(x/N)\Big)\Big(\eta_0(x)-\eta_1(x)\Big)$$
$$+\Big(\vartheta_2(x/N)-\vartheta_3(x/N)\Big)\Big(\eta_3(x)-\eta_2(x)\Big)\Bigg\},$$

and

$$\frac{\nu^N_{\hat\theta}(\xi,\sigma^x\omega)}{\nu^N_{\hat\theta}(\xi,\omega)}=\exp\Bigg\{\Big(\vartheta_1(x/N)-\vartheta_3(x/N)\Big)\Big(\eta_3(x)-\eta_1(x)\Big)$$
$$+\Big(\vartheta_2(x/N)-\vartheta_0(x/N)\Big)\Big(\eta_0(x)-\eta_2(x)\Big)\Bigg\}.$$

□

5.B Quadratic variations computations

We compute here the quadratic variations of the two jump martingales appearing in Section 5.5. Using computations as in C. Cocozza and C. Kipnis [14],

Lemma 5.B.1. *For $t\geqslant 0$, $1\leqslant i\leqslant 3$, $1\leqslant j\leqslant d$ such that $x,x+e_j\in\Lambda_N$, $\widetilde J^{x,x+e_j}_t(\eta_i)=J^{x,x+e_j}_t(\eta_i)-N^2\int_0^t\eta_{i,s}(x)(1-\eta_{i,s}(x+e_j))ds$ and $\widetilde J^{x+e_j,x}_t(\eta_i)=J^{x+e_j,x}_t(\eta_i)-N^2\int_0^t\eta_{i,s}(x+e_j)(1-\eta_{i,s}(x))ds$ are two $\mathbb{P}^N_{\mu_N}$-martingales whose quadratic variations are given by*

$$\langle \widetilde J^{x,x+e_j}(\eta_i)\rangle_t=N^2\int_0^t\eta_{i,s}(x)(1-\eta_{i,s}(x+e_j))ds \tag{5.B.1}$$

$$\langle \widetilde J^{x+e_j,x}(\eta_i)\rangle_t=N^2\int_0^t\eta_{i,s}(x+e_j)(1-\eta_{i,s}(x))ds \tag{5.B.2}$$

Proof. Consider jumps over the bond $(x,x+e_k)$, by writing the generator of diffusion as in (5.3.9), we shall decompose the jumps associated to the exchanges of particles between each type i and j, $i,j\in\{0,1,2,3\}$. That is,

$$J^{x,x+e_1}(\eta_i)=\sum_{j\neq i}J^{x,x+e_1}_{i\to j}(\xi,\omega)\quad\text{and}\quad J^{x,x+e_1}(\eta_i)=\sum_{j\neq i}J^{x,x+e_1}_{j\leftarrow i}(\xi,\omega).$$

where for fixed i, $J^{x,x+e_1}_{i\to j}$ correspond to the exchanges of particles over the bond $(x,x+e_1)$ when x is in state i and $x+e_1$ is in state j.

For $z, z+e_k \in \Lambda_N$, consider the function $f^{z,z+e_k}_{i\to j}(\xi,\omega) = \eta_3(z)\eta_1(z+e_k)$. Then,

$$\begin{aligned}
\mathcal{L}_N f_z(\eta_i) &= \sum_{\substack{x,y\in\Lambda_N\\ \|x-y\|=1}} \Big(\eta_j^{x,y}(z)\eta_i^{x,y}(z+e_k) - \eta_j(z)\eta_i(z+e_k)\Big)\\
&= \sum_{\substack{u\in\Lambda_N\\ \|u-z\|=1, u\neq z+e_k}} \Big(\eta_j(u)\eta_i(z+e_k) - \eta_j(z)\eta_i(z+e_k)\Big)\\
&\quad + \Big(\eta_j(z+e_k))\eta_i(z) - \eta_j(z)\eta_i(z+e_k)\Big)\\
&\quad + \sum_{\substack{v\in\Lambda_N\\ \|v-(z+e_k)\|=1, v\neq z}} \Big(\eta_j(z)\eta_i(v) - \eta_j(z)\eta_i(z+e_k)\Big)
\end{aligned}$$

The martingale problem states that

$$\widetilde{f}^{i\to j}_{z,z+e_1}(\xi_t,\omega_t) := f^{i\to j}_{z,z+e_1}(\xi_t,\omega_t) - \int_0^t \mathcal{L}_N f^{i\to j}_{z,z+e_1}(\xi_s,\omega_s)ds$$

is a $\mathbb{P}^N_{\mu_N}$-martingale. Consider the predictable process $g^{i\to j}_{z,z+e_1}(\xi_s,\omega_s) = \eta_{i,s-}(z)\eta_{j,s-}(z+e_1))$. Since the set $\{s : \eta_{i,s-}(z)\eta_{j,s-}(z+e_k \neq \eta_{i,s}(z)\eta_{j,s}(z+e_k)\}$ is ds-negligible,

$$\begin{aligned}
&\int_0^t g^{i\to j}_{z,z+e_k}(\xi_s,\omega_s)d\widetilde{f}^{i\to j}_{z,z+e_k}(\xi_s,\omega_s)\\
&= \int_0^t g^{i\to j}_{z,z+e_k}(\xi_s,\omega_s)df^{i\to j}_{z,z+e_k}(\xi_s,\omega_s) - \int_0^t g^{i\to j}_{z,z+e_k}(\xi_s,\omega_s)\mathcal{L}_N\Big(f^{i\to j}_{z,z+e_k}(\xi_s,\omega_s)ds\\
&= \sum_{s\leqslant t} g^{i\to j}_{z,z+e_k}(\xi_s,\omega_s)\Big(\eta_{j,s}(z)\eta_{i,s}(z+e_k) - \eta_{j,s-}(z)\eta_{i,s-}(z+e_k)\Big)\\
&\quad - \int_0^t \Bigg\{ \sum_{\substack{u\in\Lambda_N\\ \|u-z\|=1, u\neq z+e_k}} \Big(\eta_{j,s}(u)\eta_{i,s}(z+e_k) - \eta_{j,s}(z)\eta_{i,s}(z+e_k)\Big)\eta_{i,s}(z)\eta_{j,s}(z+e_k))\\
&\quad - \Big(\eta_{j,s}(z+e_k))\eta_{i,s}(z) - \eta_{j,s}(z)\eta_{i,s}(z+e_k)\Big)\eta_{i,s}(z)\eta_{j,s}(z+e_k))\\
&\quad - \sum_{\substack{v\in\Lambda_N\\ \|v-(z+e_k)\|=1, v\neq z}} \Big(\eta_{j,s}(z)\eta_{i,s}(v) - \eta_{j,s}(z)\eta_{i,s}(z+e_k)\Big)\eta_{i,s}(z)\eta_{j,s}(z+e_k)\Bigg\}ds\\
&= J^{i\to j}_{z,z+e_j}(\xi_t,\omega_t) - \int_0^t \eta_{i,s}(z)\eta_{j,s}(z+e_k)ds.
\end{aligned}$$

Let $V^{i\to j}_{z,z+e_k}(\xi_t,\omega_t) = \int_0^t \mathcal{L}_N f^{i\to j}_{z,z+e_k}(\xi_s,\omega_s)ds$. By Itô's lemma,

$$\widetilde{f}^{i\to j}_{z,z+e_k}(\xi_t,\omega_t)f^{i\to j}_{z,z+e_k}(\xi_t,\omega_t) \\ = \int_0^t \Big(f^{i\to j}_{z,z+e_k}(\xi_s,\omega_s)\widetilde{f}^{i\to j}_{z,z+e_k}(\xi_s,\omega_s) + \int_0^t \widetilde{f}^{i\to j}_{z,z+e_k}(\xi_s,\omega_s)f^{i\to j}_{z,z+e_k}(\xi_s,\omega_s)ds.$$

Therefore, $\widetilde{f}^{i\to j}_{z,z+e_k}(\xi_t,\omega_t)V^{i\to j}_{z,z+e_k}(\xi_t,\omega_t) - \int_0^t \Big(\widetilde{f}^{i\to j}_{z,z+e_k}(\xi_s,\omega_s)V^{i\to j}_{z,z+e_k}(\xi_s,\omega_s)ds$ is a martingale and

$$\Big(\widetilde{f}^{i\to j}_{z,z+e_k}(\xi_t,\omega_t)\Big)^2 = \widetilde{f}^{i\to j}_{z,z+e_k}(\xi_t,\omega_t) - 2\Big(\widetilde{f}^{i\to j}_{z,z+e_k}(\xi_t,\omega_t)V^{i\to j}_{z,z+e_k}(\xi_t,\omega_t) \\ - \int_0^t \Big(\widetilde{f}^{i\to j}_{z,z+e_k}(\xi_s,\omega_s)dV^{i\to j}_{z,z+e_k}(\xi_s,\omega_s)\Big) + \int_0^t (1 - f^{i\to j}_{z,z+e_k}(\xi_s,\omega_s)dV^{i\to j}_{z,z+e_k}(\xi_s,\omega_s)ds.$$

By Doob's decomposition, $\Big\langle \widetilde{f}^{i\to j}_{z,z+e_k}(\xi,\omega)\Big\rangle_t = \int_0^t \Big(1 - 2f^{i\to j}_{z,z+e_k}(\xi_s,\omega_s)\Big)dV^{i\to j}_{z,z+e_k}(\xi_s,\omega_s)ds.$ Hence, since $\int_0^t \Big(g^{i\to j}_{z,z+e_k}(\xi_s,\omega_s)d\widetilde{f}^{i\to j}_{z,z+e_k}(\xi_s,\omega_s)ds = \widetilde{J}^{i\to j}_{z,z+e_k}(\xi_t,\omega_t)$,

$$\Big\langle \widetilde{J}^{i\to j}_{z,z+e_k}(\xi,\omega)\Big\rangle_t = \int_0^t \Big(\eta_{i,s}(z)\eta_{j,s}(z+e_k)\Big)^2\Big(1 - f^{i\to j}_{z,z+e_k}(\xi_s,\omega_s)\Big)dV^{i\to j}_{z,z+e_k}(\xi_s,\omega_s)ds \\ = \int_0^t \eta_{i,s}(z)\eta_{j,s}(z+e_k)\Big)dV^{i\to j}_{z,z+e_k}(\xi_s,\omega_s) \\ = \int_0^t \eta_{i,s}(z)\eta_{j,s}(z+e_k)ds$$

By inverting the direction of the jumps, we compute similarly that

$$\Big\langle \Big(\widetilde{J}^{i\leftarrow j}_{z,z+e_k}(\xi,\omega)\Big)\Big\rangle_t = \int_0^t \eta_{j,s}(z)\eta_{i,s}(z+e_k)\Big)ds$$

□

Lemma 5.B.2. *For $t \geqslant 0$, $1 \leqslant i \leqslant 3$ and $x \in \Lambda_N$, $\widetilde{Q}^x_t(\eta_i) = Q^x_t(\eta_i) - \int_0^t \tau_x f_i(\xi_s,\omega_s)ds$ is a $\mathbb{P}^N_{\mu_N}$-martingale whose quadratic variations is given by*

Proof. As in previous lemma, one shall decompose the non-conservative dynamics according to interaction between each type of particles $i, j \in \{0,1,2,3\}$. That i,s

$$Q^x_t(\eta_i) = \sum_{j\neq i} Q^{i\to j}_z(\xi,\omega) - Q^{j\leftarrow i}_z(\xi,\omega)$$

where for fixed i, $Q^{i\to j}_z(\xi,\omega)$ corresponds to state j when z is in state i and $Q^{i\leftarrow j}_z(\xi,\omega)$ corresponds to flips to state i when z is in state j. It suffices to consider the case $i = 0, j = 1$ as others follow a similar way.

As in the proof of Lemma 5.B.1, for $z \in \Lambda_N$ consider $f^{1\leftarrow 0}_z(\xi_t,\omega_t) = \eta_{1,s}(z)$ and $g^{1\leftarrow 0}_z(\xi_t,\omega_t) = \eta_{0,s-}(z)$. Identical computations give

$$\Big\langle \Big(Q^{1\leftarrow 0}_z(\xi,\omega)\Big)\Big\rangle_t = \int_0^t \beta(z,\xi,\omega)\eta_{s,0}(z)ds.$$

To conclude the case $Q_t^x(\eta_1)$, compute as well

$$\langle Q_z^{1\leftarrow 3}(\xi,\omega)\rangle_t = \int_0^t \eta_{3,s}(z)ds,\ \langle Q_z^{1\to 0}(\xi,\omega)\rangle_t = \int_0^t \eta_{s,1}(z)ds,$$
$$\langle Q_z^{1\to 3}(\xi,\omega)\Big)\rangle_t = \int_0^t r\eta_{s,1}(z)ds.$$

□

5.C Estimates in bounded domain

Lemma 5.C.1. *For a smooth profile $\widehat{\theta} : \overline{\Lambda} \to (0,1)^3$ such that $\theta\big|_\Gamma = \widehat{b}$, there exist positive constants A_0, A_0' and A_1 depending only on $\widehat{\theta}$ such that for any $c > 0$, for any $f \in L^2(\nu_{\widehat{\theta}}^N)$,*

$$\langle L_{\widehat{b},N} f, f\rangle = -D_N^{\widehat{b}}(f^2), \tag{5.C.1}$$
$$\langle \mathcal{L}_N f, f\rangle = -A_0\mathcal{D}_N^0(f^2) + A_0' N^{d-2}\|f\|^2_{L^2(\nu_{\widehat{\theta}}^N)}, \tag{5.C.2}$$
$$\langle \mathbb{L}_N f, f\rangle = A_1 N^d \|f\|^2_{L^2(\nu_{\widehat{\theta}}^N)}. \tag{5.C.3}$$

Proof. Since $\nu_{\widehat{\theta}}^N$ is reversible with respect to the generator $L_{\widehat{b},N}$, (5.C.1) is immediate. To prove (5.C.2), remark that for all $A, B, c > 0$, $A(B-A) = -(B-A)^2 + B(B-A)$ and use (5.3.17)

$$\begin{aligned}
\langle \mathcal{L}_N f, f\rangle &= \sum_{x,y\in\Lambda_N} \int f(\xi,\omega)\Big(f(\xi^{x,y},\omega^{x,y}) - f(\xi,\omega)\Big)d\nu_{\widehat{\theta}}^N(\xi,\omega)\\
&= -\frac{1}{2}D_N^0(f) + \frac{1}{2}\sum_{x,y\in\Lambda_N}\int f(\eta^{x,y})(f(\eta) - f(\eta^{x,y}))d\nu_{\widehat{\theta}}^N(\xi,\omega)\\
&\quad + \frac{1}{2}\sum_{x,y\in\Lambda_N}\int f(\eta)\Big(f(\eta^{x,y}) - f(\eta)\Big)d\nu_{\widehat{\theta}}^N(\xi,\omega)\\
&= -\frac{1}{2}D_N^0(f) + \frac{1}{2}\sum_{x,y\in\Lambda_N}\sum_{i,j}\Big(f(\eta) - f(\eta^{x,y})\Big)f(\eta)R_{i,j}^{x,y}(\widehat{\theta})d\nu_{\widehat{\theta}}^N(\xi,\omega)\\
&\leqslant -(\frac{1}{2} - \frac{1}{4c})D_N^0(f^2) + \frac{c}{2}N^{d-2}\|f\|^2_{L^2(\nu_{\widehat{\theta}}^N)} + O(\frac{1}{N^2})
\end{aligned}$$

where we did a Taylor expansion of $R_{i,j}^{x,y}(\widehat{\theta})$ which was defined in 5.3.8.

$$\begin{aligned}
\langle \mathbb{L}_N f, f\rangle &= I_1 + I_2\\
&= \sum_{x\in\Lambda_N}\int \Big(\beta_N(x,\xi,\omega)(1-\xi(x)) + \xi(x)\Big)f(\xi,\omega)\Big(f(\sigma^x\xi,\omega) - f(\xi,\omega)\Big)d\nu_{\widehat{\theta}}^N(\xi,\omega)
\end{aligned}$$

$$+\sum_{x\in\Lambda_N}\int\Big(r(1-\omega(x))+\omega(x)\Big)f(\xi,\omega)\Big(f(\xi,\sigma^x\omega)-f(\xi,\omega)\Big)d\nu_{\widehat{\theta}}^N(\xi,\omega)$$

Let us deal with the first integral, the second will follow the same way. Since all the rates are bounded, we have

$$\begin{aligned}I_1 &\leqslant C(\lambda_1,\lambda_2,r)\sum_{x\in\Lambda_N}\int\Big(f(\xi,\omega)f(\sigma^x\xi,\omega)-f(\xi,\omega)^2\Big)d\nu_{\widehat{\theta}}^N(\xi,\omega)\\ &\leqslant C(\lambda_1,\lambda_2,r)\sum_{x\in\Lambda_N}\int\Big(\frac{c}{2}f(\xi,\omega)^2+\frac{1}{2c}f(\sigma^x\xi,\omega)^2-f(\xi,\omega)^2\Big)d\nu_{\widehat{\theta}}^N(\xi,\omega)\end{aligned}$$

for an arbitrary $c>0$ with use (5.3.17) for the last inequality. Choosing $c=2$,

$$\begin{aligned}I_1 &\leqslant \frac{C(\lambda_1,\lambda_2,r)}{4}\sum_{x\in\Lambda_N}\int f(\sigma^x\xi,\omega)^2 d\nu_{\widehat{\theta}}^N(\xi,\omega)\\ &\leqslant \frac{C(\lambda_1,\lambda_2,r)B(\widehat{\theta})}{4}\sum_{x\in\Lambda_N}\int f(\xi,\omega)^2 d\nu_{\widehat{\theta}}^N(\xi,\omega)\end{aligned}$$

for some positive constant $B(\widehat{\theta})$ depending on $\widehat{\theta}$ through a change of variables related to Lemma 5.A.2(ii). Similarly, one gets

$$I_2 \leqslant \frac{C(\lambda_1,\lambda_2,r)B'(\widehat{\theta})}{4}\sum_{x\in\Lambda_N}\int f(\xi,\omega)^2 d\nu_{\widehat{\theta}}^N(\xi,\omega)$$

for some positive constant $B'(\widehat{\theta})$ from a change of variables corresponding to Lemma 5.A.2(iii). Since $f\in L^2(\nu_{\widehat{\theta}})$, we have

$$\langle \mathbb{L}_N f, f\rangle \leqslant A_1 N^d \|f\|^2_{L^2(\nu_{\widehat{\theta}}^N)}.$$

with $A_1=\dfrac{C(\lambda_1,\lambda_2,r)B(\widehat{\theta})}{4}+\dfrac{C(\lambda_1,\lambda_2,r)B'(\widehat{\theta})}{4}$. □

Perspectives

So far, we have been concerned with a competition model for a population dynamics with random environment. Our results proved the existence of a unique phase transition on $\mathbb{Z}^d$ within a dynamic random environment on one hand, and survival and extinction conditions on $\mathbb{Z}$ within a quenched random environment on the other hand. Assuming these stochastic dynamics are underlying a microscopic scale, the hydrodynamic equation of the system with stirring is given by a non-linear reaction-diffusion system, with additionally Dirichlet boundary conditions when in presence of stochastic reservoirs.

The following is an overview of possible guidelines.

Weak survival. Let $\mathcal{T}_d$ be the homogeneous tree whereby each vertex has $d+1$ neighbours. A particular property that belongs to the basic contact process is that it exhibits two phase transitions on $\mathcal{T}_d$, meaning that according to Definitions (1.2) of Chapter 1, λ_c and λ_s are distinct. Following works on percolation by G. Grimmett and C. Newman [36], R. Pemantle [68] proved that in dimension 3, weak survival occurs and

$$\lambda_c < \lambda_s,\ \lambda_c \leqslant \frac{1}{d-1},\ \lambda_s \geqslant \frac{1}{2\sqrt{d}}.$$

Extensions to dimension 2 and inhomogeneous trees were done by T.M. Liggett [56, 55] and A. Stacey [73]. Still close to percolation behaviours [36], R. Durrett and R. Schinazi [24] proved the existence of infinitely many invariant measure in the intermediate phase.

See R. Schinazi [69, Chapter VII], T.M. Liggett [57, Part I.4] for further details on the contact process on the tree.

The existence of a weak survival arose interests in investigating the behaviour of the process within the intermediate phase. Biologically, a weak survival phase is thought of as being the tipping phase where the SIT program would fail or success.

Some observations lead to think the behaviour of our symmetric multitype process is similar to the basic contact process. Though, D. Griffeath showed that weak survival can occur for *totally asymmetric contact processes* on $\mathbb{Z}$.

Random environment. We studied the contact process in a particular quenched random environment. Improved results would rely on finding conditions on the distribution of the environment for the survival or extinction of the process such as in C. Newman and S. Volchan [66] did in a 1-dimension case. Primarily based on percolation techniques, they proved the survival of the process with conditions on the tail of distribution of the environment, when the growth rate is small enough.

Studying the hydrodynamics of our system, it is foreseeable to investigate the process in the presence of a macroscopic random environment or disorder.

Stirring limits and Predator-prey systems. By scaling and stirring the particle system in Chapters 4 and 5, we proved it converges to the solution of a reaction-diffusion system. As stated by R. Durrett [19, Chapter 9], it seems if one gets enough information on the limiting differential system, one would be able to derive the existence of stationary distributions for the system with stirring.

Here, we studied a system evolving in a bulk in contact with stochastic reservoirs, creating a flow of particles through the volume. The macroscopic system has been investigated in a more intricate way than the microscopic one used to be. Going back to a microscopic scale, thus to the dynamics of population, it is relevant to ask ourselves how it alters the survival and extinction phases of the process.

Hydrostatics. In finite volume, e.g. when $\Lambda_N = \{-N, \cdots, N\} \times \mathbb{T}_N^{d-1}$, the Markov process $(\xi_t, \omega_t)_{t \geqslant 0}$ on Λ_N is irreducible : for each $N \geqslant 1$, there exists a unique invariant measure μ_N^{stat}. In this case, we may derive the hydrostatic limit of the system.

References

[1] E. D. Andjel. Survival of multidimensional contact process in random environments. *Bol. Soc. Brasil. Mat. (N.S.)*, 23(1-2) :109–119, 1992.

[2] D. J. Barsky, G. R. Grimmett, and C. M. Newman. Percolation in half-spaces : equality of critical densities and continuity of the percolation probability. *Probab. Theory Related Fields*, 90(1) :111–148, 1991.

[3] L. Bertini, A. De Sole, D. Gabrielli, G. Jona-Lasinio, and C. Landim. Large deviations of the empirical current in interacting particle systems. *Teor. Veroyatn. Primen.*, 51(1) :144–170, 2006.

[4] L. Bertini, C. Landim, and M. Mourragui. Dynamical large deviations for the boundary driven weakly asymmetric exclusion process. *Ann. Probab.*, 37(6) :2357–2403, 2009.

[5] C. Bezuidenhout and G. R. Grimmett. The critical contact process dies out. *Ann. Probab.*, 18(4) :1462–1482, 1990.

[6] C. Bezuidenhout and G. R. Grimmett. Exponential decay for subcritical contact and percolation processes. *Ann. Probab.*, 19(3) :984–1009, 1991.

[7] P. Billingsley. *Convergence of probability measures*. Wiley Series in Probability and Statistics : Probability and Statistics. John Wiley & Sons Inc., New York, second edition, 1999. A Wiley-Interscience Publication.

[8] T. Bodineau and B. Derrida. Current large deviations for asymmetric exclusion processes with open boundaries. *J. Stat. Phys.*, 123(2) :277–300, 2006.

[9] T. Bodineau and M. Lagouge. Large deviations of the empirical currents for a boundary-driven reaction diffusion model. *Ann. Appl. Probab.*, 22(6) :2282–2319, 2012.

[10] D. Borrello. Stochastic order and attractiveness for particle systems with multiple births, deaths and jumps. *Electron. J. Probab.*, 16 :no. 4, 106–151, 2011.

[11] M. Bramson and R. Durrett. A simple proof of the stability criterion of Gray and Griffeath. *Probab. Theory Related Fields*, 80(2) :293–298, 1988.

[12] M. Bramson, R. Durrett, and R. H. Schonmann. The contact process in a random environment. *Ann. Probab.*, 19(3) :960–983, 1991.

[13] H. Brezis. *Functional analysis, Sobolev spaces and partial differential equations*. Universitext. Springer, New York, 2011.

[14] C. Cocozza and C. Kipnis. Existence de processus Markoviens pour des systèmes infinis de particules. *Ann. Inst. H. Poincaré Sect. B (N.S.)*, 13(3) :239–257, 1977.

[15] A. De Masi and E. Presutti. *Mathematical methods for hydrodynamic limits*, volume 1501 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1991.

[16] R. L. Dobrushin. Prescribing a system of random variables by conditional distributions. *Theory.Probab.Appl.*, 15 :458–486 (1970), 1970.

[17] R. Durrett. Oriented percolation in two dimensions. *Ann. Probab.*, 12(4) :999–1040, 1984.

[18] R. Durrett. *Lecture notes on particle systems and percolation*. The Wadsworth & Brooks/Cole Statistics/Probability Series. Wadsworth & Brooks/Cole Advanced Books & Software, Pacific Grove, CA, 1988.

[19] R. Durrett. Ten lectures on particle systems. In *Lectures on probability theory (Saint-Flour, 1993)*, volume 1608 of *Lecture Notes in Math.*, pages 97–201. Springer, Berlin, 1995.

[20] R. Durrett. Special invited paper : coexistence in stochastic spatial models. *Ann. Appl. Probab.*, 19(2) :477–496, 2009.

[21] R. Durrett and S. Levin. The importance of being discrete (and spatial). *Theor. Pop. Biol.*, 46 :363–394, 1994.

[22] R. Durrett and S. A. Levin. Stochastic spatial models : a user's guide to ecological applications. *Philosophical Transactions of the Royal Society of London. Series B : Biological Sciences*, 343(1305) :329–350, 1994.

[23] R. Durrett and C. Neuhauser. Coexistence results for some competition models. *Ann. Appl. Probab.*, 7(1) :10–45, 1997.

[24] R. Durrett and R. Schinazi. Intermediate phase for the contact process on a tree. *Ann. Probab.*, 23(2) :668–673, 1995.

[25] R. Durrett and R. B. Schinazi. Asymptotic critical value for a competition model. *Ann. Appl. Probab.*, 3(4) :1047–1066, 1993.

[26] R. Durrett and G. Swindle. Are there bushes in a forest ? *Stochastic Process. Appl.*, 37(1) :19–31, 1991.

[27] V. Dyck, J. Hendrichs, and A. Robinson. *Sterile insect technique : Principles and Practice in area-wide integrated pest management*. The Wadsworth & Brooks/Cole Statistics/Probability Series. Springer, Pacific Grove, CA, 2005.

[28] J. Farfan, C. Landim, and M. Mourragui. Hydrostatics and dynamical large deviations of boundary driven gradient symmetric exclusion processes. *Stochastic Process. Appl.*, 121(4) :725–758, 2011.

[29] C. M. Fortuin, P. W. Kasteleyn, and J. Ginibre. Correlation inequalities on some partially ordered sets. *Comm. Math. Phys.*, 22 :89–103, 1971.

[30] O. Garet and R. Marchand. The critical branching random walk in a random environment dies out. *Electron. Commun. Probab.*, 18 :no. 9, 15, 2013.

[31] T. Gobron and E. Saada. Couplings, attractiveness and hydrodynamics for conservative particle systems. *Ann. Inst. Henri Poincaré Probab. Stat.*, 46(4) :1132–1177, 2010.

[32] D. Griffeath. *Additive and cancellative interacting particle systems*, volume 724 of *Lecture Notes in Mathematics*. Springer, Berlin, 1979.

[33] G. R. Grimmett. *Percolation*, volume 321 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, second edition, 1999.

[34] G. R. Grimmett. *Probability on graphs*, volume 1 of *Institute of Mathematical Statistics Textbooks*. Cambridge University Press, Cambridge, 2010. Random processes on graphs and lattices.

[35] G. R. Grimmett and J. M. Marstrand. The supercritical phase of percolation is well behaved. *Proc. Roy. Soc. London Ser. A*, 430(1879) :439–457, 1990.

[36] G. R. Grimmett and C. M. Newman. Percolation in $\infty + 1$ dimensions. In *Disorder in physical systems*, Oxford Sci. Publ., pages 167–190. Oxford Univ. Press, New York, 1990.

[37] M. Z. Guo, G. C. Papanicolaou, and S. R. S. Varadhan. Nonlinear diffusion limit for a system with nearest neighbor interactions. *Comm. Math. Phys.*, 118(1) :31–59, 1988.

[38] T. E. Harris. Nearest-neighbor Markov interaction processes on multidimensional lattices. *Advances in Math.*, 9 :66–89, 1972.

[39] T. E. Harris. Contact interactions on a lattice. *Ann. Probab.*, 2 :969–988, 1974.

[40] T. E. Harris. Additive set-valued Markov processes and graphical methods. *Ann. Probability*, 6(3) :355–378, 1978.

[41] R. Holley and T. M. Liggett. The survival of contact processes. *Ann. Probability*, 6(2) :198–206, 1978.

[42] C. Kipnis and C. Landim. *Scaling limits of interacting particle systems*, volume 320 of *Grundlehren der Mathematischen Wissenschaften*. Springer-Verlag, Berlin, 1999.

[43] C. Kipnis, S. Olla, and S. R. S. Varadhan. Hydrodynamics and large deviation for simple exclusion processes. *Comm. Pure Appl. Math.*, 42(2) :115–137, 1989.

[44] W. Klassen and C. Curtis. *History of the Sterile Insect Technique*. Springer Netherlands, 2005.

[45] A. Klein. Extinction of contact and percolation processes in a random environment. *Ann. Probab.*, 22(3) :1227–1251, 1994.

[46] E. F. Knipling. Possibilities of insect control or eradication through the use of sexually sterile males. *J. Econ. Entomol.*, 48(4) :459–462, 1955.

[47] N. Konno. *Phase transitions of interacting particle systems*. World Scientific, 1994.

[48] N. Konno, R. B. Schinazi, and H. Tanemura. Coexistence results for a spatial stochastic epidemic model. *Markov Process. Related Fields*, 10(2) :367–376, 2004.

[49] K. Kuoch. Phase transition for a contact process with random slowdowns. *(in preparation)*.

[50] K. Kuoch, M. Mourragui, and E. Saada. Hydrodynamic limits for a generalized contact process with stochastic reservoirs or in infinite volume. *(in preparation)*.

[51] C. Landim and M. Mourragui. Hydrodynamic limit of mean zero asymmetric zero range processes in infinite volume. *Ann. Inst. H. Poincaré Probab. Statist.*, 33(1) :65–82, 1997.

[52] T. M. Liggett. Spatially inhomogeneous contact processes. In *Spatial stochastic processes*, volume 19 of *Progr. Probab.*, pages 105–140. Birkhäuser Boston, Boston, MA, 1991.

[53] T. M. Liggett. The survival of one-dimensional contact processes in random environments. *Ann. Probab.*, 20(2) :696–723, 1992.

[54] T. M. Liggett. Improved upper bounds for the contact process critical value. *Ann. Probab.*, 23(2) :697–723, 1995.

[55] T. M. Liggett. Branching random walks and contact processes on homogeneous trees. *Probab. Theory Related Fields*, 106(4) :495–519, 1996.

[56] T. M. Liggett. Multiple transition points for the contact process on the binary tree. *Ann. Probab.*, 24(4) :1675–1710, 1996.

[57] T. M. Liggett. *Stochastic interacting systems : contact, voter and exclusion processes*, volume 324 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1999.

[58] T. M. Liggett. *Interacting particle systems*. Classics in Mathematics. Springer-Verlag, Berlin, 2005. Reprint of the 1985 original.

[59] T. M. Liggett, R. H. Schonmann, and A. M. Stacey. Domination by product measures. *Ann. Probab.*, 25(1) :71–95, 1997.

[60] N. Madras, R. B. Schinazi, and R. H. Schonmann. On the critical behavior of the contact process in deterministic inhomogeneous environments. *Ann. Probab.*, 22(3) :1140–1159, 1994.

[61] R. Marra and M. Mourragui. Phase segregation dynamics for the Blume-Capel model with Kac interaction. *Stochastic Process. Appl.*, 88(1) :79–124, 2000.

[62] T. S. Mountford. A metastable result for the finite multidimensional contact process. *Canad. Math. Bull.*, 36(2) :216–226, 1993.

[63] M. Mourragui. Comportement hydrodynamique des processus de sauts, de naissances et de morts. *Thèse de Doctorat, Université de Rouen*, 1993.

[64] M. Mourragui and E. Orlandi. Lattice gas model in random medium and open boundaries : hydrodynamic and relaxation to the steady state. *J. Stat. Phys.*, 136(4) :685–714, 2009.

[65] C. Neuhauser. Ergodic theorems for the multitype contact process. *Probab. Theory Related Fields*, 91(3-4) :467–506, 1992.

[66] C. M. Newman and S. B. Volchan. Persistent survival of one-dimensional contact processes in random environments. *Ann. Probab.*, 24(1) :411–421, 1996.

[67] K. Oelschläger. A law of large numbers for moderately interacting diffusion processes. *Z. Wahrsch. Verw. Gebiete*, 69(2) :279–322, 1985.

[68] R. Pemantle. The contact process on trees. *Ann. Probab.*, 20(4) :2089–2116, 1992.

[69] R. B. Schinazi. *Classical and spatial stochastic processes*. Birkhäuser Boston, Inc., Boston, MA, 1999.

[70] F. Spitzer. Random processes defined through the interaction of an infinite particle system. In *Probability and Information Theory (Proc. Internat. Sympos., McMaster Univ., Hamilton, Ont., 1968)*, pages 201–223. Springer, Berlin, 1969.

[71] F. Spitzer. Interaction of Markov processes. *Advances in Math.*, 5 :246–290 (1970), 1970.

[72] H. Spohn. *Large scale dynamics of interacting particle systems.* Texts and Monographs in Physics, Springer-Verlag, Heidelberg, 1991.

[73] A. M. Stacey. The existence of an intermediate phase for the contact process on trees. *Ann. Probab.*, 24(4) :1711–1726, 1996.

[74] J. E. Steif and M. Warfheimer. The critical contact process in a randomly evolving environment dies out. *ALEA Lat. Am. J. Probab. Math. Stat.*, 4 :337–357, 2008.

[75] J. van den Berg, G. R. Grimmett, and R. B. Schinazi. Dependent random graphs and spatial epidemics. *Ann. Appl. Probab.*, 8(2) :317–336, 1998.

[76] A. Van der Vloedt and W. Klassen. The development and application of the sterile insect technique (sit) for new world screwworm eradication. *World Animal Review*, pages 42–49, 1991.

[77] C. Zuily. *Éléments de distributions et d'équations aux dérivées partielles. Cours et problèmes résolus.* Collection Sciences Sup, Editions Dunod, Paris, 2002.